A TREATISE

ON

HYDROSTATICS AND HYDRODYNAMICS.

BY

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CAMBRIDGE:
DEIGHTON, BELL, AND CO.
LONDON: BELL AND DALDY.
1859.

Cambridge:
PRINTED BY C. J. CLAY, M.A.
AT THE UNIVERSITY PRESS.

PREFACE.

In compiling the present treatise, I have endeavoured to place before the reader the course of study, in theoretical Hydrostatics and Hydrodynamics, which is usually required in the Examination for the Mathematical Tripos.

For the main portions of the subject, I have consulted chiefly Poisson's *Mécanique* and Duhamel's *Cours de Mécanique*, but I have occasionally found it necessary to refer to the larger and more important works of Laplace and Lagrange, the *Mécanique Céleste* and the *Mécanique Analytique*. The problem discussed in Chapter VII., for instance, is given with greater clearness and fulness by Laplace, than by any subsequent writer whose works I have been able to consult.

By the kindness of Professor Stokes I have been permitted to make some extracts, on a difficult part of the subject, from a very valuable paper by him in the Cambridge Philosophical Transactions.

The Examples by which the various Chapters are illustrated, and which it is hoped will form a sufficient and useful set of exercises for the student, have been chosen almost entirely from the Senate-House papers of the last few years, and from the Examination papers of St John's College and Caius College.

The investigations, relating to the vibrations of rods and strings, which have been introduced in Chapter XIII., can hardly be said to belong to the province of Hydrodynamics; they are however so closely connected with the theory of sound, and

especially of musical sounds, that a chapter on the subject is not complete without them, and I have therefore ventured to devote a few articles to their discussion.

I have to offer my best thanks to several friends who have kindly assisted me by their advice, and in particular to Mr G. D. Liveing, of St John's College, and also to the Rev. J. R. Lunn, of St John's College, to whom I am indebted for important hints and corrections in the chapter on musical sounds.

W. H. BESANT.

ST JOHN'S COLLEGE,
June 7, 1859.

CONTENTS.

CHAPTER I.

CHAPTER II.

CHAPTER III.

CHAPTER IV.

CHAPTER V.

CHAPTER VI.

CHAPTER VII.

CHAPTER VIII.

CHAPTER IX.

CHAPTER X.

CHAPTER XI.

CHAPTER XII.

CHAPTER XIII.

APPENDIX.

ERRATA.

Page 22. Ex. 2, lines 1 and 8, *for* n *read* $n-1$.

Page 23, line 1, *for* $\frac{\mu g}{n+1}$ *read* $\frac{\mu g}{n}$, with corresponding corrections in the next three lines.

HYDROSTATICS.

CHAPTER I.

1. We learn from common experience that such substances as air and water are characterised by the ease with which portions of their mass can be removed, and by their extreme divisibility. These properties are illustrated by various common facts; if, for instance, we consider the ease with which fluids can be made to permeate each other, the extreme tenuity to which one fluid can be reduced by mixture with a large portion of another fluid, the rarefaction of air which can be effected by means of an air-pump, and other facts of a similar kind, it is clear that, practically, the divisibility of fluid is unlimited: we find, moreover, that in separating portions of fluids from each other, the resistance offered to the division is very slight, and in general almost inappreciable. By a generalization from such observations, the conception naturally arises of a substance possessing in the highest degree these properties, which exist, in a greater or less degree, in every fluid with which we are acquainted, and hence we are led to the following

Definition of a Fluid.

2. *A fluid is an aggregation of particles which yield to the slightest effort made to separate them from each other.*

If then an indefinitely thin plane be made to divide a fluid in any direction, no resistance will be offered to the division, and the pressure exerted by the fluid on the plane will be entirely normal to it; that is, a perfect fluid is assumed to have no "viscosity," no property of the nature of friction.

The following fundamental property of a fluid is therefore obtained from the above definition.

The pressure of a fluid is always normal to any surface with which it is in contact.

As a matter of fact, all fluids do more or less offer a resistance to separation or division, but, just as the idea of a rigid body is obtained from the observation of bodies in nature which only change form slightly on the application of great force, so is the idea of a perfect fluid obtained from our experiences of substances which possess the characteristics of extremely easy separability and apparently unlimited divisibility.

3. Fluids are divided into Liquids and Gases; the former, such as water and mercury, are not sensibly compressible, except under very great pressures; the latter are easily compressible, and expand freely if permitted to do so.

Hence the former are sometimes called inelastic, and the latter elastic fluids.

4. Fluids are acted upon by the force of gravity in the same way as solids; with regard to liquids this is obvious; and that air has weight can be shewn directly by weighing a closed vessel, exhausted as far as possible: moreover, the phænomena of the tides shew that fluids are subject to the attractive forces of the sun and moon as well as of the earth, and it is assumed, from these and other similar facts, that fluids of all kinds are subject to the law of gravitation, that is, that they attract, and are attracted by, all other portions of matter, in accordance with that law.

Measure of the Pressure of Fluids.

5. Consider a mass of fluid at rest under the action of any forces, and let A be the area of a plane surface exposed to the action of the fluid, that is, in contact with it, and P the force which is required to counterbalance the action of the fluid upon A. If the action of the fluid upon A be uniform, then $\frac{P}{A}$ is the pressure on each unit of the area A. If the pressure be not uniform, it must be considered as varying continuously from point to point of the area A, and if ϖ be the pressure on a small portion α of the area about a given point, then $\frac{\varpi}{\alpha}$ will approximately express the *rate* of pressure over α. When α is indefinitely diminished let $\frac{\varpi}{\alpha}$ ultimately $=p$, then p is defined to be

the measure of the pressure at the point considered, p being the pressure which would be exerted on an unit of area, if the rate of pressure over the unit were uniform and the same as at the point considered.

The pressure upon any small area α about a point, the pressure *at* which is p, is therefore $p\alpha + \gamma$, where γ vanishes ultimately in comparison with $p\alpha$ when α (and consequently $p\alpha$) vanishes.

6. In order to employ the principles of Statics in the discussion of the equilibrium of fluids, the following proposition is necessary.

In a mass of fluid at rest any portion may be supposed to become solid without any other change in the circumstances of the equilibrium.

For, if this supposition be made, there will be no alteration in the forces acting on the fluid, and the action between the solidified portion and the rest of the fluid, or between the solidified portion and any smooth surface with which it may be in contact, will be, as before, normal to its surface; the equilibrium of the solid can therefore be considered as maintained by the external forces which act upon it, and the pressure of the remaining fluid.

7. *The pressure at any point of a fluid at rest is the same in every direction.*

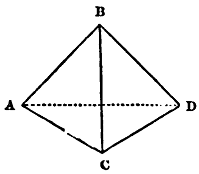
This is the most important of the characteristic properties of a fluid; it can be deduced from Articles (2) and (6) in the following manner:

Let a small tetrahedron of fluid be supposed solidified; then it is kept at rest by the pressures on its faces, and by the impressed force on its mass.

The former forces depending on the areas of the faces vary as the square, and the latter depending on the volume and density varies as the cube of one of the edges of the solid, which is considered to be homogeneous, and therefore supposing the solid indefinitely diminished, while it retains always a similar form, the latter force vanishes in comparison with the pressures

on the faces; and these pressures consequently form a system of forces in equilibrium.

Let p, p' be the units of pressure on the faces ABC, BCD, and resolve the forces parallel to the edge AD; then, since the projections of the areas ABC, BCD on a plane perpendicular to AD are the same (each equal to α suppose), we have ultimately,



$$p\alpha = p'\alpha,$$

$$\text{or} \quad p = p'.$$

And similarly it may be shewn that the pressures on the other two faces are each equal to p or p'.

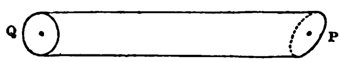
As the tetrahedron may be taken with its faces in any direction, it follows that the pressure at a point is the same in every direction.

8. The following proof of the foregoing proposition is taken from Cauchy's *Exercices**.

Let P and Q be two points in a fluid at a finite distance from each other; about PQ as axis describe a cylinder of very small radius, draw a plane through Q perpendicular to QP, draw any plane through P, and suppose the portion of fluid PQ to become solid.

The solid PQ is kept at rest by the pressures on its ends and on its curved surface, and by the impressed forces which act upon it.

Let p, p' be the pressures at Q and P, α the area of the section Q of the cylinder, and α' of the section P; then the pressure $p'\alpha'$ on the end P, resolved parallel to the axis of the cylinder, is equal to $p'\alpha$, and therefore



$$p'\alpha - p\alpha = \text{the impressed force, resolved parallel to } QP.$$

Now whatever be the direction of the plane through P, this impressed force, when the radius of the cylinder is indefinitely

* *Seconde Année*, 1827, page 23.

diminished, is ultimately equal to the impressed force on the portion QP of the cylinder cut off by a plane through P perpendicular to the axis*, that is, to

$$\int_0^{PQ} f\rho\alpha\, dx,$$

where mf is the force on a particle m of the fluid at a distance x from Q. Hence

$$p' = p + \int_0^{PQ} \rho f dx,$$

or p' is constant for all positions of the plane through P.

Transmission of Fluid Pressure.

9. *Any pressure, or additional pressure, applied to the surface, or to any other part, of an incompressible fluid kept at rest, is transmitted equally to all parts of the fluid.*

This property of incompressible fluids is a direct result of experiment, and, as such, is sometimes assumed. It is however deducible from our definition of a fluid by aid of the proposition of Art. 6.

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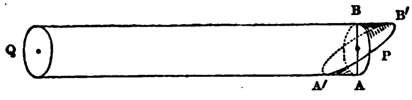
* The following considerations may complete this part of the proof:

Let AB, $A'B'$ be the two planes through P; $\rho\,\rho'$ the mean densities of APA', BPB'; and ff' the accelerations of the forces which are acting on these portions of fluid.



Then the difference of the forces on QAB and $QA'B'$, (the volumes of which are equal)

$$= \text{the difference of the forces on } APA' \text{ and } BPB'$$
$$= (\rho' f' - \rho f) \,.\, \text{vol} \,.\, APA'$$
$$= \delta(\rho f) \,.\, \frac{2}{3\pi}\, aAA',$$

and therefore $p' = p + \int_0^{QP} \rho f dx + \frac{2}{3\pi} AA' \,.\, \delta(\rho f)$.

The forces being continuous, the last term is obviously evanescent compared with the other quantities in the equation, and p' is therefore constant.

Let P be a point in the surface of a fluid at rest, and Q any other point in the fluid; about the straight line PQ describe a cylinder of very small radius, bounded by the surface at P and by a plane through Q, perpendicular to QP, and suppose this cylinder to become solid.

If the pressure at P be increased by p, the additional force on the cylinder, resolved in the direction of its axis, is $p\alpha$, α being the area of the section of the cylinder perpendicular to its axis, and this must be counteracted by an equal force $p\alpha$ at Q in the direction QP, since the pressure of the fluid on the curved surface is perpendicular to the axis. The pressure at Q is therefore increased by p.

If the straight line PQ do not lie entirely in the fluid, P and Q can be connected by a number of straight lines, all lying in the fluid, and a repetition of the above reasoning will shew that the pressure p is transmitted, unchanged, to the point Q.

10. In consequence of this property, a mass of inelastic fluid can be used as a 'machine' for the purpose of multiplying power.

Thus, if in a closed vessel full of water two apertures be made and pistons A, A' fitted in them, any pressure P applied to one piston must be counteracted by a pressure P' on the other piston, such that $P' : P$ in the ratio of the areas $A' : A$, for the increased rate of pressure at every point of A is transmitted to *every* point of A', and the amount of pressure upon A' depends therefore upon its area*.

The action between the two is analogous to the action of a lever, and it is clear that by increasing A' and diminishing A, we can make the ratio $P' : P$ as large as we please. It is equally clear that this is independent of the quantity of fluid employed, and hence

"*The Hydrostatic Paradox.*"

"Any quantity of fluid, however small, can be made to support any weight, however large."

This statement is of course only practically true within the

* Bramah's Press is an instance of the practical use of this property of fluid.

limits assigned by the strength of the materials employed in the construction of the necessary machinery.

11. The pressure of an elastic fluid is found to depend upon its density and temperature, as well as upon the nature of the fluid itself.

When the temperature is constant, experiment shews that the pressure varies inversely as the space occupied by the fluid, that is, directly as its density.

Hence if ρ be the density of an elastic fluid, and p its pressure, then, as long as the temperature remains the same,

$$p = k\rho,$$

where k is a constant, to be determined experimentally in any given case.

Measures of Weight, Mass, and Density.

12. The weight, mass and density of a fluid are measured in the same way as for solid bodies.

If W be the weight of a mass M of fluid, then in accordance with the usual convention which defines the unit of mass,

$$W = Mg.$$

Next, let ρ be the density and V the volume of the mass M of fluid, which we suppose homogeneous, and ρ' the density, V' the volume of an unit of mass of a standard fluid;

Then $\qquad M : 1 :: \rho V : \rho' V'.$

Let $V' = 1$, and make ρ' the unit of density.

$$\text{Then} \quad M = \rho V,$$

$$\text{and} \quad W = g\rho V.$$

13. Sometimes the conception of the *intrinsic* weight of a fluid is required, and for this the term "specific gravity" is employed.

Measure of Specific Gravity.

The specific gravity of a fluid is the ratio of the weight of any volume of it to that of an equal volume of a standard substance.

Thus, if w be the weight of an unit of volume of the standard substance, and s the specific gravity of any fluid, the weight W of a volume V is given by the equation

$$W = swV,$$

or if $w = 1$,

$$W = sV.$$

14. It will be observed that the unit of weight, implied in the equation $W = sV$, is not necessarily the same as in the equation $W = g\rho V$, and in fact, from the arbitrary way in which the units have been chosen, they are in general different quantities.

In order to illustrate this point, we proceed to consider, in two cases, the nature of the relations between the several units employed in the equations under discussion.

First case. Let the unit of length be the same in both equations, and take also the standard substance the same; then the equation $W = sV$ implies that the unit of weight is the weight of an unit of volume of the standard substance, whereas from $W = g\rho V$, putting $\rho = 1$ and $V = 1$, we obtain

Weight of an unit of volume of the standard substance $= g$ times the unit of weight,

and therefore the unit of weight in the latter formula is $\frac{1}{g}$th of the unit in the former.

Second case. Let the unit of length be as before the same in both equations, and suppose the unit of weight assigned, and the same in both, taking it for example to be 1 lb. Now it is known that a cubic foot of water at a certain temperature weighs 1000 oz.; if then one foot be the unit of length, and s represent the specific gravity of water, the equation $W = sV$ gives

$$\frac{1000}{16}\text{lbs.} = s\,\text{lbs.}, \quad \text{or } s = \frac{1000}{16},$$

that is, the specific gravity of water $= \frac{1000}{16}$ (the specific gravity of the standard substance).

Again, considering ρ as the density of water, we obtain from the equation $W = g\rho V$,

$$\frac{1000}{16}\text{lbs.} = g\rho\text{lbs.}, \quad \text{or } \rho = \frac{1000}{16g},$$

that is, the density of the standard substance : the density of water :: $16g$: 1000; the numerical value of g of course depending on the unit of time selected.

In a similar manner other arrangements of the units can be treated, but the two cases just discussed will sufficiently illustrate the meanings of the symbols employed.

15. In the previous articles no account has been taken of fluids in which the density is variable; but it is easy to conceive the density of a mass of inelastic fluid varying continuously from point to point, and it will be hereafter found that a mass of elastic fluid, at rest under the action of gravity, and having a constant temperature throughout, is necessarily heterogeneous: the density *at* a point of a fluid must therefore be measured in the same way as the pressure at a point, or any other continuously varying quantity.

Measure of the density at any point of a heterogeneous mass of fluid.

Let m be the mass of a volume v of fluid enclosing a given point, and suppose ρ the density of a homogeneous fluid such that the mass of a volume v is equal to m, or such that

$$m = \rho v;$$

then ρ may be defined as the mean density of the portion v of the heterogeneous fluid, and the ultimate value of ρ when v is indefinitely diminished, supposing it always to enclose the point, is the density of the fluid at that point.

In a similar manner the specific gravity at any point of a heterogeneous fluid is measured.

CHAPTER II.

THE CONDITIONS OF THE EQUILIBRIUM OF FLUIDS.

16. TAKING the most general case, suppose a mass of fluid, elastic or non-elastic, homogeneous or heterogeneous, to be at rest under the action of given forces, and let it be required to determine the conditions of equilibrium, and the pressure at any point.

Let x, y, z, be the co-ordinates referred to rectangular axes, of any point P in the fluid, and let Q be a point near it, so taken that PQ is parallel to the axis of x.

Take $x + \delta x$, y, z, as the co-ordinates of Q; about PQ describe a small prism or cylinder bounded by planes perpendicular to PQ, and conceive this cylinder to be solidified.

Let α be the area of the section of the cylinder perpendicular to its axis, p the pressure at P and $p + \delta p$ the pressure at Q.

Then, α being very small the pressure at any point of the plane P will be very nearly equal to p, and the pressure upon it will therefore be

$$(p + \gamma)\, \alpha,$$

where γ vanishes in comparison with p when α is indefinitely diminished.

We can therefore consider α so small that γ may be neglected in comparison with p, and the pressure on the end P of the cylinder may be taken equal to $p\alpha$, and similarly the pressure on the end Q equal to

$$(p + \delta p)\, \alpha.$$

If ρ be the mean density of the cylinder PQ, its mass $= \rho\alpha\delta x$, and $X\rho\alpha\delta x$ will represent the force on PQ parallel to its axis,

if $X\delta m$, $Y\delta m$, $Z\delta m$, be the components of the forces acting on a particle δm of fluid at the point xyz.

Hence for the equilibrium of PQ, a necessary condition is

$$(p + \delta p)\,\alpha - p\alpha = X\rho\alpha\delta x,$$

or

$$\delta p = \rho X dx.$$

Proceeding to the limit when δx, and therefore δp, is indefinitely diminished, ρ will be the density at P, and we obtain

$$\frac{dp}{dx} = \rho X^*.$$

By a similar process,

$$\frac{dp}{dy} = \rho Y,$$

$$\frac{dp}{dz} = \rho Z.$$

But

$$dp = \frac{dp}{dx}\,dx + \frac{dp}{dy}\,dy + \frac{dp}{dz}\,dz;$$

$$\therefore\ dp = \rho\,(Xdx + Ydy + Zdz) \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (\alpha),$$

the equation which determines the pressure.

17. It is therefore an essential condition of equilibrium that $\rho\,(Xdx + Ydy + Zdz)$ should be a perfect differential of some function $f(x, y, z)$; and

$$\left.\begin{aligned} \therefore\ \frac{d}{dx}(\rho Y) &= \frac{d}{dy}(\rho X) \\ \frac{d}{dy}(\rho Z) &= \frac{d}{dz}(\rho Y) \\ \frac{d}{dz}(\rho X) &= \frac{d}{dx}(\rho Z) \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (\beta),$$

* In the above proof, α is taken so small that its linear dimensions may be neglected in comparison with δx; that is, the change in p, corresponding to a change δx in x, is considered, undisturbed by any alterations in y and z.

CHAPTER II.

THE CONDITIONS OF THE EQUILIBRIUM OF FLUIDS.

16. Taking the most general case, suppose a mass of fluid, elastic or non-elastic, homogeneous or heterogeneous, to be at rest under the action of given forces, and let it be required to determine the conditions of equilibrium, and the pressure at any point.

Let x, y, z be the co-ordinates referred to rectangular axes, of any point P in the fluid, and let Q be a point near it, so taken that PQ is parallel to the axis of x.

Take $x + \delta x$, y, z, as the co-ordinates of Q: about PQ describe a small prism or cylinder bounded by planes perpendicular to PQ, and conceive this cylinder to be solidified.

Let α be the area of the section of the cylinder perpendicular to its axis, p the pressure at P and $p + \delta p$ the pressure at Q.

Then, α being very small the pressure at any point of the plane P will be very nearly equal to p, and the pressure upon [illegible] will therefore be

$$(p + \eta)\,\alpha,$$

where η vanishes in comparison with p when α is indefi[illegible] diminished.

We can therefore consider α so small that η may be [illegible] in comparison with p, and the pressure on the end [illegible] cylinder may be taken equal to $p\alpha$, and similarly the [illegible] the end Q equal to

$$(p + \delta p)\,\alpha.$$

If ρ be the [illegible] and $X\rho\alpha\delta x$ will [illegible]

from which by differentiating, multiplying the equations respectively by Z, X, and Y, and adding, we obtain

$$X\left(\frac{dY}{dz}-\frac{dZ}{dy}\right)+Y\left(\frac{dZ}{dx}-\frac{dX}{dz}\right)+Z\left(\frac{dX}{dy}-\frac{dY}{dx}\right)=0, \ldots\ldots (\gamma),$$

a necessary although not a sufficient condition of equilibrium.

18. If the forces tend to fixed centres and are functions of the distances from those centres, we have

$$X=\Sigma\left\{\phi(r)\frac{x-a}{r}\right\},\quad Y=\Sigma\left\{\phi(r)\frac{y-b}{r}\right\},\quad Z=\Sigma\left\{\phi(r)\frac{z-c}{r}\right\},$$

when (abc) are co-ordinates of the centre to which the force $\phi(r)$ tends.

Now $$r^2=(x-a)^2+(y-b)^2+(z-c)^2,$$

$$\therefore\ Xdx+Ydy+Zdz=\Sigma\phi(r)\,dr,$$

and $$dp=\rho\,\Sigma\{\phi(r)\}dr.$$

In this case, since

$$\frac{dX}{dy}=\Sigma\left\{\phi'(r)\frac{x-a}{r}\frac{y-b}{r}-\phi(r)\frac{x-a}{r^2}\frac{y-b}{r}\right\},$$

and $$\frac{dY}{dx}=\Sigma\left\{\phi'(r)\frac{y-b}{r}\frac{x-a}{r}-\phi(r)\frac{y-b}{r^2}\frac{x-a}{r}\right\},$$

it is obvious that the equation (γ) is always satisfied, but it is not to be inferred that the equilibrium of a heterogeneous fluid is always possible with such a system of forces.

When the density is constant, the equations (β) become

$$\frac{dX}{dy}=\frac{dY}{dx},\quad \frac{dZ}{dy}=\frac{dY}{dz},\quad \frac{dX}{dz}=\frac{dZ}{dx},$$

which are in this case always satisfied, and therefore the equilibrium of a homogeneous fluid under the action of such forces is always possible.

19. If the fluid be elastic, an additional condition is introduced, for if the temperature be constant,

$$p = k\rho;$$

$$\therefore \frac{dp}{p} = \frac{1}{k}(Xdx + Ydy + Zdz) \ldots\ldots\ldots\ldots (\delta).$$

When the forces tend to fixed centres and are functions of the distances, Art. (18), this equation takes the form

$$\frac{dp}{p} = \frac{1}{k}\Sigma\phi(r)\,dr,$$

and p can be determined.

If the temperature be variable, the relation between the pressure density and temperature is found to be

$$p = k\rho(1 + \alpha t),$$

where t is the temperature, and $\alpha = {\cdot}003665$.

In this case $\dfrac{dp}{p} = \dfrac{1}{k(1 + \alpha t)}(Xdx + Ydy + Zdz)$,

and therefore t must be a function of x, y, and z.

20. From equation (α), if the fluid be homogeneous and

$$Xdx + Ydy + Zdz = dV,$$

$$p = \rho V + C.$$

If the fluid be heterogeneous and ρ a function of x, y, and z, such that

$$\rho(Xdx + Ydy + Zdz) = dU,$$

$$p = U + C,$$

and from (δ) if there be equilibrium

$$k \log \frac{p}{C} = V,$$

$$\therefore p = C\epsilon^{\frac{V}{k}}, \text{ and } \rho = \frac{C}{k}\epsilon^{\frac{V}{k}},$$

the constant being obtained from the particular circumstances of the equilibrium.

In any of these cases, if the pressure at any particular point be given the constant can be determined.

In the last case if the mass of fluid, and the space within which it is contained be given, the constant is determined.

21. In all cases, in which the equilibrium of the fluid is possible we obtain by integration

$$p = \phi(x, y, z)$$

If p be constant and equal to p',

$$\phi(x, y, z) = p' \dots\dots\dots\dots\dots\dots\dots\dots (A),$$

is the equation to the surface at all points of which the pressure is constant, and by giving different values to p' we obtain a series of *surfaces of equal pressure*, and the external surface, or free surface, is obtained by making p' equal to the pressure external to the fluid.

If the external pressure be zero the free surface is therefore

$$\phi(x, y, z) = 0.$$

22. The quantities

$$\frac{d\phi}{dx}, \frac{d\phi}{dy}, \text{ and } \frac{d\phi}{dz},$$

which are proportional to the direction cosines of the normal at the point (x, y, z) of the surface A, are equal to

$$\frac{dp}{dx}, \frac{dp}{dy}, \frac{dp}{dz},$$

respectively, i.e. to ρX, ρY, ρZ, and are therefore proportional to X, Y, Z.

Hence the resultant force at any point is in direction of the normal to the surface of equal pressure passing through the point*.

* This last result may also be obtained in the following manner:

Consider two consecutive surfaces of equal pressure, containing between them a stratum of fluid, and let a small circle be described about a point P in one surface, and a portion of the fluid cut out by normals through the circumference. The portion of fluid so cut out may be considered rigid, and kept at rest by the impressed force, and the pressures on its ends and on its circumference. Being very nearly a small cylinder, and the pressures on all points of its circumference equal, the difference of pressures on its two faces must be due to the force, which must therefore act in the same direction as these pressures, *i.e.* in direction of the normal at P.

23. In the particular cases in which $Xdx + Ydy + Zdz$ is a perfect differential dV, ρ must be a function of V.

$$\text{For } dp = \rho dV$$

and dp being a perfect differential, ρ must be a function of V;

$$\text{let } \rho = f'(V)$$

$$\text{then } dp = f'(V)\,dV,$$

$$p = f(V) + C.$$

Hence V, and therefore ρ is a function of p, and surfaces of equal pressure are also surfaces of equal density.

If the fluid be elastic and the temperature variable

$$\frac{dp}{p} = \frac{1}{k(1+\alpha t)} dV.$$

Hence by a similar process of reasoning t is a function of p, and surfaces of equal pressure are also surfaces of equal temperature.

24. If however $Xdx + Ydy + Zdz$ be not a perfect differential these surfaces will not in general coincide.

1st. Let the fluid be heterogeneous and incompressible; then the surfaces of equal pressure and of equal density are given respectively by the equations

$$dp = 0, \quad d\rho = 0,$$

$$\text{or} \quad \left.\begin{aligned} Xdx + Ydy + Zdz = 0 \\ \frac{d\rho}{dx}dx + \frac{d\rho}{dy}dy + \frac{d\rho}{dz}dz = 0 \end{aligned}\right\} \dots\dots\dots\dots\dots\dots\text{(B)}.$$

These then are the differential equations of surfaces which by their intersections determine curves of equal pressure and density.

From (B) we obtain

$$\frac{dx}{Z\frac{d\rho}{dy} - Y\frac{d\rho}{dz}} = \frac{dy}{X\frac{d\rho}{dz} - Z\frac{d\rho}{dx}} = \frac{dz}{Y\frac{d\rho}{dx} - X\frac{d\rho}{dy}} \dots\dots\dots\text{(C)}.$$

But from the conditions of equilibrium we have

$$\rho\frac{dX}{dy}+X\frac{d\rho}{dy}=\rho\frac{dY}{dx}+Y\frac{d\rho}{dx},$$

$$\rho\frac{dY}{dz}+Y\frac{d\rho}{dz}=\rho\frac{dZ}{dy}+Z\frac{d\rho}{dy},$$

$$\rho\frac{dZ}{dx}+Z\frac{d\rho}{dx}=\rho\frac{dX}{dz}+X\frac{d\rho}{dz},$$

and therefore the equations (C) become

$$\frac{dx}{\frac{dZ}{dy}-\frac{dY}{dz}}=\frac{dy}{\frac{dX}{dz}-\frac{dZ}{dx}}=\frac{dz}{\frac{dY}{dx}-\frac{dX}{dy}}\text{............(D)},$$

The differential equations of the curves of equal pressure and density.

2nd. Let the fluid be elastic and of variable temperature.

Then $\dfrac{dp}{p}=\dfrac{1}{k(1+\alpha t)}(Xdx+Ydy+Zdz)$,

and the curves of equal pressure and temperature are given by the simultaneous equations

$$dp=0;\quad dt=0;$$

or

$$\left.\begin{aligned}Xdx+Ydy+Z\,dz&=0\\ \frac{dt}{dx}dx+\frac{dt}{dy}dy+\frac{dt}{dz}dz&=0\end{aligned}\right\}.$$

But since $\dfrac{dp}{p}$ is a perfect differential the conditions of equilibrium are in this case

$$\frac{d}{dy}\left(\frac{X}{1+\alpha t}\right)=\frac{d}{dx}\left(\frac{Y}{1+\alpha t}\right),$$

$$\text{or}\quad(1+\alpha t)\frac{dX}{dy}-\alpha X\frac{dt}{dy}=(1+\alpha t)\frac{dY}{dx}-\alpha Y\frac{dt}{dx},$$

with similar equations between X and Z, and Y and Z respectively.

From the preceding equations, we obtain

$$\frac{dx}{Y\frac{dt}{dz}-Z\frac{dt}{dy}}=\frac{dy}{Z\frac{dt}{dx}-X\frac{dt}{dz}}=\frac{dz}{X\frac{dt}{dy}-Y\frac{dt}{dx}},$$

$$X\frac{dt}{dy}-Y\frac{dt}{dx}=\frac{1+\alpha t}{\alpha}\left(\frac{dX}{dy}-\frac{dY}{dx}\right)$$

$$Z\frac{dt}{dx}-X\frac{dt}{dz}=\frac{1+\alpha t}{\alpha}\left(\frac{dZ}{dx}-\frac{dX}{dz}\right)$$

$$Y\frac{dt}{dz}-Z\frac{dt}{dy}=\frac{1+\alpha t}{\alpha}\left(\frac{dY}{dz}-\frac{dZ}{dy}\right);$$

and $\therefore$ $$\frac{dx}{\frac{dZ}{dy}-\frac{dY}{dx}}=\frac{dy}{\frac{dX}{dz}-\frac{dZ}{dx}}=\frac{dz}{\frac{dY}{dx}-\frac{dX}{dy}},$$

equations of the same form as (D), are in this case the differential equations of the curves of equal pressure and temperature, and therefore also of equal density.

25. As a particular case, consider the equilibrium of a fluid under the action of gravity, which may be considered to act in a constant direction if the fluid do not extend over a large portion of the earth's surface.

In this case $X=0,\quad Y=0,\quad Z=-g,$

if z be measured vertically upwards;

$$\therefore\ dp=-g\rho dz.$$

Therefore if the fluid be inelastic, and homogeneous,

$$p=-g\rho z+C.$$

If the fluid be elastic, $p=k\rho$,

$$\text{and } \therefore\ \frac{dp}{p}=-\frac{g}{k}dz;$$

$$\therefore \log \frac{p}{C} = -\frac{g}{k} z,$$

$$p = C\epsilon^{-\frac{g}{k}z}.$$

In each case then it follows that the surfaces of equal pressure are horizontal planes, and therefore the surface of *liquid* at rest under the action of gravity is a horizontal plane.

If the fluid be inelastic, and the origin at a depth h below the surface, the pressure at which $= \Pi$,

$$p = \Pi + g\rho\,(h - z),$$

or, if $\Pi = 0$, $p = g\rho$ (the depth below the surface).

26. In the case of an inelastic heterogeneous fluid at rest under the action of gravity, then, measuring z downwards, the equation

$$dp = g\rho dz,$$

shews that ρ must be a function of z.

The density and pressure are therefore constant for all points in the same horizontal plane.

As an example, let $\rho \propto z^n = \mu z^n$,

$$\text{then } p = g\mu \frac{z^{n+1}}{n+1} + \Pi.$$

27. As a second example, let a given volume V of incompressible fluid be acted upon by forces

$$-\frac{\mu x}{a^2}, \; -\frac{\mu y}{b^2}, \; -\frac{\mu z}{c^2},$$

respectively parallel to the axes;

$$\therefore dp = \rho\left(-\frac{\mu x}{a^2}\,dx - \frac{\mu y}{b^2}\,dy - \frac{\mu z}{c^2}\,dz\right),$$

$$p = C - \frac{\mu\rho}{2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right).$$

The surfaces of equal pressure are therefore similar ellipsoids, and the equation to the free surface is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{2C}{\mu\rho},$$

assuming that there is no external pressure.

The condition which determines the constant is that the volume of the fluid is given, and we have

$$V = \frac{4}{3}\pi abc \,.\left(\frac{2C}{\mu\rho}\right)^{\frac{3}{2}},$$

$$\text{and}\quad C = \frac{\mu\rho}{2}\,.\left(\frac{3V}{4\pi abc}\right)^{\frac{2}{3}}.$$

Rotating Fluid.

28. If a quantity of fluid revolve uniformly, and without any relative displacement of its particles, about a fixed axis, the preceding equations will enable us to determine the pressure at any point, and the nature of the surfaces of equal pressure.

For, in such cases of relative equilibrium, every particle of the fluid moves uniformly in a circle, and the resultant of the external forces acting on any particle m of the fluid, and of the fluid pressure upon it, must be equal to a force $m\omega^2 r$ *towards* the axis, ω being the angular velocity, and r the distance of m from the axis; it follows therefore that the external forces, combined with the fluid pressures, and forces $m\omega^2 r$ acting *from* the axis, form a system in statical equilibrium, to which the equations of the previous articles are applicable.

29. *A mass of homogeneous liquid, contained in a vessel, revolves uniformly about a vertical axis; required to determine the pressure at any point, and the surfaces of equal pressure.*

Take the vertical axis as the axis of z; then, resolving the force $m\omega^2 r$ parallel to the axes, its components are $m\omega^2 x$ and $m\omega^2 y$, and the general equation of fluid equilibrium becomes

$$dp = \rho\,(\omega^2 x\,dx + \omega^2 y\,dy - g\,dz),$$

and therefore

$$p = \rho\,\{\tfrac{1}{2}\omega^2 (x^2 + y^2) - gz\} + C.$$

The surfaces of equal pressure are therefore paraboloids of revolution, and if the vessel be open at the top, the free surface is given by the equation

$$\omega^2 (x^2 + y^2) - 2gz + \frac{2C}{\rho} = \frac{2\Pi}{\rho},$$

where Π is the external pressure.

The constant must be determined by help of the data of each particular case.

For instance, let the vessel be closed at the top and be *very nearly* filled with fluid, and let $\Pi = 0$; then taking the origin at the highest point of the axis, $p = 0$, when x y and z vanish, and therefore $C = 0$, and

$$p = \rho \left\{\tfrac{1}{2}\omega^2 (x^2 + y^2) - gz\right\}.$$

30. Next consider the case of elastic fluid enclosed in a vessel which rotates about a vertical axis; as before

$$dp = \rho \left\{\omega^2 (xdx + ydy) - gdz\right\},$$

$$\text{and } \; p = k\rho;$$

$$\therefore \; k \log \rho = \omega^2 \frac{x^2 + y^2}{2} - gz + C,$$

so that the surfaces of equal pressure and density are paraboloids.

Let the containing vessel be a cylinder rotating about its axis, and suppose the whole mass of fluid given; then, to determine the constant, consider the fluid arranged in elementary horizontal rings, each of uniform density: let r be the radius of one of these rings, at a height z, δr its horizontal and δz its vertical thickness, h the height, and a the radius of the cylinder;

$$\text{the mass of the ring} = 2\pi\rho r\, \delta r\, \delta z,$$

$$\text{and the whole mass } (M) \text{ of the fluid} = \int_0^a \int_0^h 2\pi\rho r\, dr\, dz.$$

The origin being taken at the base of the cylinder.

$$\text{Now } \; \rho = \epsilon^{\frac{C}{k}} . \epsilon^{\frac{\omega^2 r^2 - 2gz}{2k}};$$

$$\text{and } \; \therefore \; M = \frac{2\pi k^2}{g\omega^2} \epsilon^{\frac{C}{k}} \left(\epsilon^{\frac{\omega^2 a^2}{2k}} - 1\right) \left(1 - \epsilon^{-\frac{gh}{k}}\right),$$

an equation by which C is determined.

31. In general the equation of equilibrium for a fluid revolving uniformly, and acted upon by forces of any kind, is

$$dp = \rho \left\{Xdx + Ydy + Zdz + \omega^2 (xdx + ydy)\right\}.$$

In order that the equilibrium may be possible, three equations of condition must be satisfied, expressing that dp is a perfect differential, and from these equations it will be found that the same relation* must exist between the quantities X, Y, Z, as if there were no rotation. If the conditions referred to are satisfied, the surfaces of equal pressure, and, in certain cases, the free surface can be determined; but it must be observed that a free surface is not always possible. In fact, in order that there may be a free surface, the surfaces of equal pressure must be symmetrical with respect to the axis of rotation.

EXAMPLE. *A closed vessel is completely filled with homogeneous liquid, which is made to rotate uniformly about an axis inclined to the vertical, required to find the surfaces of equal pressure.*

Let α be the inclination of the axis to the vertical, and take the axis of x in the vertical plane through the axis of rotation; then

$$\frac{1}{\rho}dp = (\omega^2 x - g \sin\alpha)\, dx + \omega^2 y\, dy - g\cos\alpha\, dz,$$

$$\frac{p}{\rho} = \tfrac{1}{2}\omega^2 (x^2 + y^2) - gx\sin\alpha - gz\cos\alpha + C,$$

and the required surfaces are paraboloids having their common axis parallel to the axis of revolution.

It will be seen that in this case the pressure *about* any given particle of fluid varies with its position in the circle in which it is moving; in other words, a given particle of fluid passes across different surfaces of equal pressure in the course of its revolution. If the vessel be just filled without compression, the constant C can be determined: on this point some remarks will be made hereafter.

Whole Pressure.

32. DEF. *The whole pressure of a fluid on any surface with which it is in contact is the sum of the normal pressures on each of its elements.*

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* That given by the equation γ, Art. 17.

If then p be the pressure at a point of an element δS of the surface,

$$p\delta S \text{ is the pressure on the element,}$$

$$\text{and } \iint p\delta S \text{ is the whole pressure,}$$

the summation extending over the whole of the surface considered.

In the particular case of gravity being the only force in action, $p = g\rho z$, measuring z vertically downwards from the surface;

$$\text{and } \iint p\delta S = \iint g\rho z\,\delta S.$$

Let $\bar{z}$ be the depth of the centre of gravity of the *surface* S,

$$\text{then } \bar{z}\,.\iint \delta S = \iint z\delta S;$$

$$\text{and } \therefore \text{ the whole pressure} = g\rho\bar{z}S,$$

i. e. the whole pressure is equal to the weight of a cylindrical column of fluid, the height of which is $\bar{z}$, and the base a plane area equal to the area of the surface.

33. We now add some examples of the determination of whole pressure.

(1) *A hemispherical bowl filled with water.*

Let r be its radius, ρ the density of water.

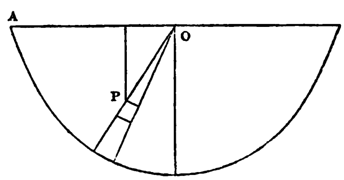
Then the surface $= 2\pi r^2$,

$$\text{and } \bar{z} = \frac{r}{2};$$

$$\therefore \text{ whole pressure} = g\rho\pi r^3,$$

i. e. whole pressure : the weight of the fluid :: 3 : 2.

(2) The density of a heavy fluid $\propto$ (depth)^n, n being a positive integer; find the whole pressure on a semicircular area immersed vertically, with its diameter in the surface.



Employing polar co-ordinates and referring to Art. (26), we see that if the density $= \mu$ (depth)^n,

the pressure at $P=\dfrac{\mu g}{n+1}r^n(\sin\theta)^n$, where $OP=r$ and $AOP=\theta$;

$\therefore$ the whole pressure $=2\displaystyle\int_0^a\int_0^{\frac{\pi}{2}}\frac{\mu g}{n+1}r^{n+1}(\sin\theta)^n\,d\theta\,dr$, if $OA=a$,

$$=\frac{2\mu g a^{n+2}}{(n+1)(n+2)}\cdot\frac{(n-1)\,.\,(n-3)\,\ldots\,1}{n\,.\,(n-2)\,\ldots\,2}\cdot\frac{\pi}{2}\ \ldots\ n \text{ even},$$

$$=\frac{2\mu g a^{n+2}}{(n+1)(n+2)}\cdot\frac{(n-1)\,\ldots\,2}{n\,\ldots\,3}\ \ldots\ n \text{ odd}.$$

(3) A cylindrical vessel is closed at the top, and very nearly filled with incompressible fluid, which rotates uniformly about the axis of the cylinder; to find the whole pressure on the curved surface and on the top of the cylinder.

In this case, taking the centre of the top as origin, and measuring z downwards,

$$\frac{p}{\rho}=\frac{\omega^2r^2}{2}+gz.$$

Let a be the radius of the cylinder, h its height; then at a depth z, the pressure at its surface

$$=\rho\left(\frac{\omega^2a^2}{2}+gz\right),$$

an element of surface $=2\pi a\,.\,\delta z$;

$\therefore$ the whole pressure on the curved surface

$$=\int_0^h 2\pi a\rho\left(\tfrac{1}{2}\omega^2a^2+gz\right)dz,$$

$$=\pi\rho a^3h\omega^2+\pi\rho agh^2.$$

The pressure on the top at a distance r from the origin $=\frac{1}{2}\rho\omega^2r^2$,

and an element of its area $=2\pi r\,\delta r$;

therefore the whole pressure on the top

$$=\int_0^a \pi\rho\omega^2r^3dr=\tfrac{1}{4}\pi\rho\omega^2a^4.$$

EXAMPLES.

1. The side AB of a triangle ABC is in the surface of a fluid, and points D, E, are taken in AC, such that the pressures on the triangles BAD, BDE, BEC, are equal; find the ratios

$$AD : DE : EC.$$

2. A triangle ABC is immersed in fluid, in such a position that the point A is in the surface and the lines AB, AC, are equally inclined to it; BC being produced to meet the surface in E, shew that the pressures on the triangles ABC, ACE, are in the ratio $AB^2 - AC^2 : AC^2$.

3. A regular tetrahedron is filled with fluid, and held so that two of its opposite edges are horizontal; compare the pressures on its several sides with the weight of the fluid.

4. A spherical mass of elastic fluid is compressed into the cube which can be inscribed within the sphere; compare the whole pressures on the surfaces of the cube and sphere.

5. A given quantity of elastic fluid is contained in a hollow sphere, and its particles are acted upon by a force to the centre of the sphere varying inversely as the distance. The sphere being supposed to vary in size, shew that the whole pressure on its surface varies inversely as its radius, provided $\mu < 3\kappa$, when μ is the absolute force, and κ the ratio of the pressure to the density of the fluid.

6. A quantity of incompressible fluid within a cylinder is acted upon by a force to a point in its axis varying directly as the distance, and is made to rotate uniformly about the axis. Taking no account of gravity, determine the nature of the free surfaces for different angular velocities; and in particular, find the angular velocity for which the free surface will be that of a cone.

7. A closed cylindrical vessel is very nearly filled with incompressible fluid, which is acted upon by a force, varying as the distance, to the middle point of the axis of a cylinder; if $2a$ be the length of the axis and c the radius of either end, shew that

the whole pressure on the curved surface : the whole pressure on the ends :: $8a^3 : 3c^3$.

Also find this ratio when the centre of force is at the centre of either end of the cylinder.

8. A vessel in the form of an inverted cone is partly filled with fluid, and closed with a lid; it is then made to revolve uniformly about its axis; if a small hole be now made at the vertex, determine how much of the fluid will escape, considering the different cases that arise according to the magnitude of the angular velocity. If this is indefinitely increased, prove that the surface of the fluid is a circular cylinder, and find its radius.

9. An open vessel, containing two fluids which do not mix, revolves round a vertical axis with a given angular velocity; find the pressure at any point of the denser fluid, when the fluids have attained a state of relative rest, the depth of the lighter fluid in that state being given.

10. A straight rod, every particle of which attracts with a force varying inversely as the square of the distance, is surrounded by a mass of homogeneous incompressible fluid; find the form of the surfaces of equal pressure.

11. A mass (M) of fluid, in which the density at any point is the sum of a given constant quantity and a quantity bearing a given constant ratio to the pressure at that point, revolves about a fixed axis with a given constant angular velocity, and is attracted to a point in that axis by a given force which varies as the distance: find the form of the free surface; and shew that its least semi-diameter (b) is determined by the equation,

$$M = m\int_0^b e^{\frac{b^2-x^2}{c^2}}\, x^2 dx,$$

when m and c are given constants.

12. A given quantity of elastic fluid is contained in a hollow sphere, and is acted upon by a force to the centre of the sphere;

find the law of force in order that the pressure on either side of a circular lamina placed with its centre at the centre of the sphere may vary as the n^{th} power of its radius, n being positive. If this be the case, prove that the pressure on the surface of the sphere $= \frac{(n+1)\,km}{a}$ where m is the mass of fluid, a the radius of the sphere, and k the ratio of the pressure to the density.

CHAPTER III.

THE RESULTANT PRESSURES OF FLUIDS ON SURFACES.

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34. IN the preceding Chapter we have shewn how to investigate the pressure *at any point* of a fluid at rest under the action of given forces; we now proceed to determine the resultants of the pressures exerted by fluids *upon surfaces* with which they are in contact.

We shall consider, first, the action of fluids on plane surfaces, secondly, of fluids under the action of gravity upon curved surfaces, and thirdly, of fluids at rest under any given forces, upon curved surfaces.

Fluid Pressures on Plane Surfaces.

35. The pressures at all points of a plane being perpendicular to it, and in the same direction, the resultant pressure is equal to the sum of these pressures, that is, to the whole pressure, and acts in the same direction.

Hence, if the fluid be incompressible and acted upon by gravity only, the resultant pressure on a plane

$$= \text{the whole pressure}$$
$$= g\rho\bar{z}A,$$

where A is the area and $\bar{z}$ the depth of the centre of gravity.

In general, if the fluid be of any kind, and at rest under the action of any given forces, take axes of x and y in the plane, and let p be the pressure at the point (x, y).

The pressure on an element of area $\delta x\delta y = p\delta x\delta y$;

$$\therefore \text{ the resultant pressure} = \iint p\,dy\,dx,$$

the integration extending over the whole of the area considered.

If polar co-ordinates be used, the resultant pressure is given by the expression

$$\iint prdrd\theta.$$

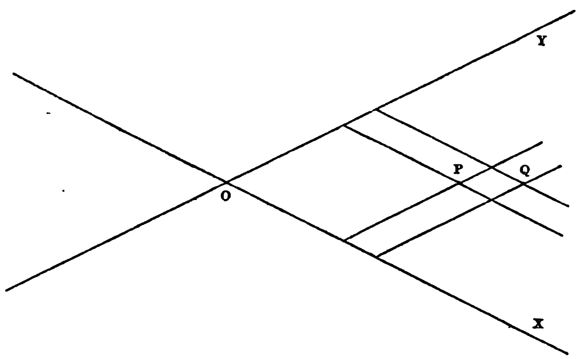
36. Def. *The centre of pressure is the point at which the direction of the single force, which is equivalent to the fluid pressures on the plane surface, meets the surface.*

The *centre of pressure* is here defined with respect to plane surfaces only; it will be seen afterwards that the resultant action of fluid on a curved surface is not always reducible to a single force.

In the case of a *heavy* fluid, it is clear that the centre of pressure of a horizontal area, the pressure on every point of which is the same, is its centre of gravity; and, since pressure increases with the depth, the centre of pressure of any plane area, not horizontal, is below its centre of gravity.

37. Prop. *To obtain formulæ for the determination of the centre of pressure.*

Let p be the pressure at the point (x, y), referred to rectangular axes in the plane, $x+\delta x$, $y+\delta y$, the co-ordinates of Q,



$\bar{x}$, $\bar{y}$, co-ordinates of the centre of pressure.

Then $\bar{y}\,.\iint pdydx$ = moment of the resultant pressure about Ox,

= the sum of the moments of the pressures on all the elements of area about Ox,

$$= \Sigma p \delta y \, \delta x \, . \, y$$

$$= \iint py dy \, dx ;$$

$$\therefore \bar{y} = \frac{\iint py dy \, dx}{\iint p dy dx},$$

and similarly $\bar{x} = \dfrac{\iint px dy \, dx}{\iint p dy \, dx}$.

The integrals being taken so as to include the area considered.

If polar co-ordinates be employed, a similar process will give the equations

$$\bar{x} = \frac{\iint pr^2 \cos\theta \, dr \, d\theta}{\iint pr \, dr \, d\theta}, \quad \bar{y} = \frac{\iint pr^2 \sin\theta \, dr \, d\theta}{\iint pr \, dr \, d\theta}.$$

38. If the fluid be homogeneous and inelastic, and if gravity be the only force in action,

$$p = g\rho h,$$

where h is the depth of the point P, below the surface, and we obtain

$$\bar{x} = \frac{\iint hx dy \, dx}{\iint h dy \, dx}, \quad \bar{y} = \frac{\iint hy \, dy \, dx}{\iint h dy \, dx} \ldots\ldots\ldots\ldots\ldots (\alpha).$$

It is sometimes useful to take for one of the axes the line of intersection of the plane with the surface of the fluid: if we take this line for the axis of x, and θ as the inclination of the plane to the horizon, $p = g\rho y \sin\theta$, and therefore

$$\bar{x} = \frac{\iint xy \, dy \, dx}{\iint y dy \, dx}, \quad \bar{y} = \frac{\iint y^2 dy \, dx}{\iint y dy \, dx} \ldots\ldots\ldots\ldots\ldots (\beta).$$

From these last equations (β) it appears that the position of the centre of pressure is independent of the inclination of the plane to the horizon, so that if a plane area be immersed in fluid, and then turned about its line of intersection with the surface at a fixed axis, the centre of pressure will remain unchanged.

If in the equations (α) we make h constant, that is, suppose the plane horizontal, $\bar{x}$ and $\bar{y}$ are the co-ordinates of the centre

of gravity of the area, a result in accordance with Art. (36); but if we suppose θ evanescent in the equations (β), the values of $\bar{x}$ and $\bar{y}$ are the same as for any finite value of the angle. This apparent anomaly is explained by considering that, however small θ be taken, the portion of fluid between the plane area and the surface of the fluid is always wedge-like in form, and the pressures at the different points of the plane, although they all vanish in the limit, do not vanish in ratios of equality, but in the constant ratios which they bear to one another for any finite value of θ*.

39. EXAMPLE (1). A given volume of inelastic fluid is at rest on a fixed plane, under the action of forces to a fixed point in the plane, varying as the distance; required to find the pressure on the plane.

Taking the fixed point as origin, the expression for the pressure at any point is

$$p = C - \tfrac{1}{2}\mu\rho\,(x^2 + y^2 + z^2) = C - \tfrac{1}{2}\mu\rho r^2,$$

where r is the distance from the origin; and if $\frac{2}{3}\pi a^3$ be the given volume, the free surface is a hemisphere of radius a, and

$$p = \tfrac{1}{2}\mu\rho\,(a^2 - r^2).$$

The portion of the plane in contact with fluid is a circle of radius a, and therefore the pressure upon it

$$= \int_0^{2\pi}\int_0^a pr\,dr\,d\theta$$
$$= \tfrac{1}{4}\pi\mu\rho a^4.$$

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* The equations of this article may be obtained by the following reasoning, which, as a slightly different method, it may be perhaps useful to insert.

Through the boundary line of the plane area draw vertical lines to the surface, and let the fluid so enclosed be considered solid; then the reaction of the plane, revolved vertically, is equal to the weight of the solidified fluid, and acts in a vertical line through its centre of gravity, and the point in which this line meets the plane is the centre of pressure.

Taking the same axes as in (38), the weight of an elementary prism, acting through the point x, y, is $g\rho h\,\delta x\,\delta y \cos\theta$ where θ is the inclination of the plane to the horizon, and therefore the *centre* of these parallel forces (Todhunter's *Statics*, Art. 66) acting at points of the plane, is given by the equations

$$\bar{x} = \frac{\iint g\rho h\,x\cos\theta\,dy\,dx}{\iint g\rho h\cos\theta\,dy\,dx},\quad \bar{y} = \frac{\iint g\rho h\,y\cos\theta\,dy\,dx}{\iint g\rho h\cos\theta\,dy\,dx},$$

or
$$\bar{x} = \frac{\iint hx\,dy\,dx}{\iint h\,dy\,dx},\quad \bar{y} = \frac{\iint hy\,dy\,dx}{\iint h\,dy\,dx}.$$

This result may be written in the form $\mu\,\frac{3}{8}a\,.\,\frac{4}{3}\pi\rho a^3$, which is the expression for the attraction on the whole mass of fluid, supposed to be condensed into a material particle at its centre of gravity, and might in fact have been at once obtained by considering the fluid solidified, and kept at rest by the attraction to the centre of force and the reaction of the plane. (See Todhunter's *Statics*, Art. 220).

Ex. 2. A rectangle has two sides horizontal, to find its centre of pressure.

Take the upper side for axis of y, and its middle point as origin; let a, b, be the sides of the rectangle, c the depth of the origin, and θ the inclination to the horizon of the plane of the rectangle.

Divide the rectangle into horizontal strips, and let x be the distance of one of these from the origin; then its depth is

$$c + x\sin\theta,$$

and the pressure on an element is

$$g\rho\,(c + x\sin\theta)\,b\delta x;$$

$$\therefore\ \bar{x} = \frac{\int_0^a x\,(c + x\sin\theta)\,dx}{\int_0^a (c + x\sin\theta)\,dx} = \frac{a}{3}\,\frac{3c + 2a\sin\theta}{2c + a\sin\theta},$$

and the value of $\bar{y}$ is evidently zero.

$$\text{If } \theta = 0,\quad \bar{x} = \frac{a}{2},\quad \text{but if } c = 0,\quad \bar{x} = \tfrac{2}{3}a,$$

results illustrative of the remarks of Art. 38.

Ex. 3. A quadrant of a circle just immersed in a heavy homogeneous fluid, with one edge in the surface.

Take Ox, the edge in the surface for axis of x,

then $p = g\rho y$,

$$\bar{x} = \frac{\int_0^a\int_0^{\sqrt{(a^2-x^2)}} xy\,dx\,dy}{\int_0^a\int_0^{\sqrt{(a^2-x^2)}} y\,dx\,dy},\qquad \bar{y} = \frac{\iint y^2\,dx\,dy}{\iint y\,dx\,dy},$$

the limits of the integrations for $\bar{y}$ being the same as for $\bar{x}$.

$$\iint y\,dx\,dy = \tfrac{1}{2}\int (a^2 - x^2)\,dx = \tfrac{1}{3}a^3,$$

$$\iint xy\,dx\,dy = \tfrac{1}{2}\int x\,.\,(a^2 - x^2)\,dx = \tfrac{1}{8}a^4,$$

$$\iint y^2\,dx\,dy = \tfrac{1}{3}\int (a^2 - x^2)^{\frac{3}{2}}\,dx = \frac{\pi a^4}{16};$$

$$\therefore\ \bar{x} = \frac{3}{8}a,\quad \bar{y} = \frac{3}{16}\pi a.$$

If the density be variable, the formulæ are

$$\bar{x} = \frac{\iint \rho xy\,dx\,dy}{\iint \rho y\,dx\,dy},\quad \bar{y} = \frac{\iint \rho y^2\,dx\,dy}{\iint \rho y\,dx\,dy}.$$

Let $\rho \propto$ depth $= \mu y$;

$$\therefore\ \bar{x} = \frac{\iint xy^2\,dx\,dy}{\iint y^2\,dx\,dy},\quad \bar{y} = \frac{\iint y^3\,dx\,dy}{\iint y^2\,dx\,dy}.$$

Now $\iint xy^2\,dx\,dy = \tfrac{1}{3}\int x\,(a^2 - x^2)^{\frac{3}{2}}\,dx = \dfrac{1}{15}a^5,$

$$\iint y^3\,dx\,dy = \tfrac{1}{4}\,(a^2 - x^2)^2\,dx = \frac{2}{15}a^5;$$

$$\therefore\ \bar{x} = \frac{16}{15}\,\frac{a}{\pi},\quad \text{and } \bar{y} = \frac{32}{15}\,\frac{a}{\pi}.$$

Employing polar co-ordinates, the last case would be treated as follows.

Taking the line in the surface as the initial line,

$$p = g\rho r \sin\theta = \mu g r^2 \sin^2\theta,$$

$$\bar{x} = \frac{\int_0^{\frac{\pi}{2}}\int_0^a r^4 \sin^2\theta \cos\theta\,dr\,d\theta}{\int_0^{\frac{\pi}{2}}\int_0^a r^3 \sin^2\theta\,dz\,d\theta} = \frac{\frac{1}{5}a^5\,.\,\frac{1}{3}}{\frac{1}{4}a^4\,.\,\frac{\pi}{4}} = \frac{16}{15}\,\frac{a}{\pi};$$

$$\bar{y} = \frac{\int_0^{\frac{\pi}{2}}\int_0^a r^4 \sin^3\theta\,dr\,d\theta}{\int_0^{\frac{\pi}{2}}\int_0^a r^3 \sin^2\theta\,dz\,d\theta} = \frac{\frac{1}{5}a^5\,.\,\frac{2}{3}}{\frac{1}{4}a^4\,.\,\frac{\pi}{4}} = \frac{32}{15}\,\frac{a}{\pi}.$$

Ex. 4. A vertical rectangle exposed to the action of the atmosphere at an equable temperature.

If Π be the atmospheric pressure at the base of the rectangle, the pressure at a height z is $\Pi\epsilon^{-\frac{gz}{k}}$, Art. (25), and if b denote the breadth, the pressure upon a horizontal strip of the rectangle

$$= \Pi\epsilon^{-\frac{gz}{k}} b dz,$$

$\therefore$ the resultant pressure, if a be the height,

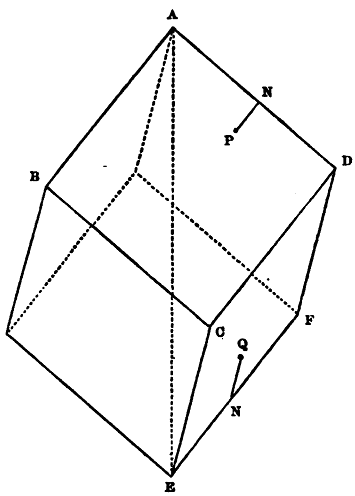
$$= \int_0^a \Pi\epsilon^{-\frac{gz}{k}} b dz = \Pi \frac{bk}{g} (1 - \epsilon^{-\frac{ga}{k}}),$$

and the height of the centre of pressure

$$= \frac{\int_0^a z\epsilon^{-\frac{gz}{k}} dz}{\int_0^a \epsilon^{-\frac{gz}{k}} dz} = \frac{k}{g} - \frac{a}{\epsilon^{\frac{ga}{k}} - 1}.$$

Ex. 5. A hollow cube is very nearly filled with fluid, and rotates uniformly about a diagonal which is vertical; required to find the pressures upon, and the centres of pressure of, its several faces.

I. For one of the upper faces $ABCD$.



Take AD, AB, as axes of x and y; r, z, the vertical and horizontal distances of any point $P(x, y)$ from A,

$$\text{then } \frac{p}{\rho} = \tfrac{1}{2}\omega^2 r^2 + gz,$$

$$z = \frac{x+y}{\sqrt{3}}, \text{ projecting the broken line } ANP \text{ on } AE,$$

$$r^2 = AP^2 - z^2 = x^2 + y^2 - z^2 = \tfrac{2}{3}(x^2 + y^2 - xy);$$

$$\therefore \text{ the pressure } (P) \text{ on } ABCD = \int_0^a \int_0^a p\,dy\,dx$$

$$= \rho \cdot \iint \left\{ \frac{\omega^2}{3}(x^2 + y^2 - xy) + \frac{g}{\sqrt{3}}(x+y) \right\} dy\,dx$$

$$= \rho \left\{ \frac{5}{36} a^4 \omega^2 + \frac{g}{\sqrt{3}} a^3 \right\}.$$

The centre of pressure is given by the equations

$$\bar{x}P = \bar{y}P = \rho \int_0^a \int_0^a x \left\{ \frac{\omega^2}{3}(x^2 + y^2 - xy) + \frac{g}{\sqrt{3}}(x+y) \right\} dy\,dx;$$

$$\therefore \bar{x} = \bar{y} = a \cdot \frac{42g + 9\sqrt{3}\omega^2 a}{72g + 10\sqrt{3}\omega^2 a}.$$

II. For one of the lower faces $ECDF$,

take EF, EC as axes, then for a point Q

$$z = a\sqrt{3} - \frac{x+y}{\sqrt{3}},$$

$$r^2 = \tfrac{2}{3}(x^2 + y^2 - xy),$$

and the rest of the process is the same as in the first case.

40. Prop. *A vessel having a plane base and plane vertical sides, contains two fluids which do not mix; to find the resultant pressures on the base and sides.*

Let ρ be the density and h the depth of the upper fluid, ρ', h', corresponding quantities for the lower fluid; the common surface

must be a horizontal plane, Art. (26) the pressure at any point of which is $g\rho h$, and the pressure at a depth z below the common surface is $g\rho h + g\rho' z$; the pressure at any point of the base is therefore $g\rho h + g\rho' h'$.

Taking b for the breadth of one of the vertical sides; the pressure of the upper fluid upon it $= \frac{1}{2} g\rho b h^2$, and the pressure of the lower fluid

$$= \int_0^{h'} g\,(\rho h + \rho' z)\, b dz = gbh'\,(\rho h + \tfrac{1}{2}\rho' h').$$

The resultant pressure is the sum of these two and is equal to

$$gb\,(\tfrac{1}{2}\rho h^2 + \rho h h' + \tfrac{1}{2}\rho' h'^2).$$

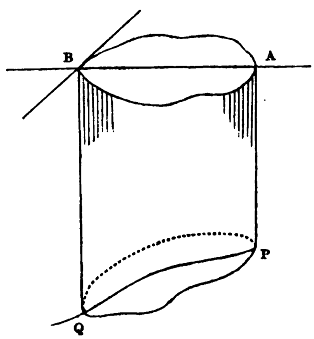
The moment of the fluid pressure on this side about its line of intersection with the surface

$$= \int_0^h g\rho b z^2 dz + \int_h^{h+h'} g\,(\rho h + \rho' z)\, bzdz:$$

performing the integrations, and dividing by the expression for the resultant pressure investigated above, we obtain the depth of the centre of pressure.

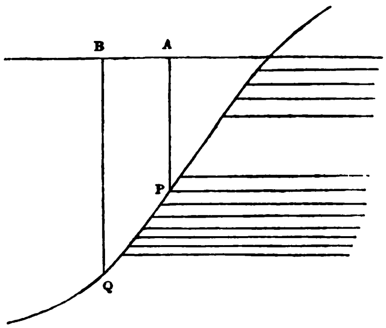
41. Prop. *To find the resultant vertical pressure on any surface of a homogeneous inelastic fluid at rest under the action of gravity.*

Let PQ be a surface exposed to the action of a heavy fluid; let AB be the projection of PQ on the surface of the fluid, and suppose the fluid contained between PQ and the vertical lines through its boundary which meet the surface in AB to be solidified.



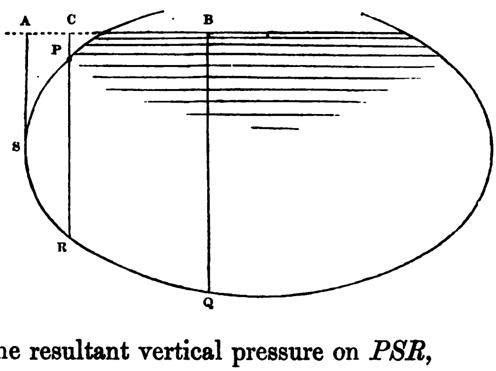
The solid AQ is supported by the horizontal pressure of the fluid, and by the reaction of PQ; this reaction resolved vertically must be equal to the weight of PQ, and conversely, the pressure on PQ is equal to the weight of AQ, and acts through its centre of gravity.

If PQ be pressed upwards by the fluid as in the figure, produce the surface of the fluid, project PQ on it as before, suppose the space AQ to be filled with fluid of the same kind, and remove the fluid from the inside.



Then the pressures at all points of PQ are the same as before, but in the contrary direction, and since the vertical pressure in this hypothetical case is equal to the weight of AQ, it follows that in the actual case, the resultant vertical pressure upwards is equal to the weight of AQ.

If the surface be pressed partially upwards and partially downwards, draw through P the highest point of the portion of surface considered a vertical plane PR, and let ACB be the projection on the surface of PSQ.



Then the resultant vertical pressure on PSR,

= the weight of the fluid in PSR,

and on RQ = ... CQ,

. Art. (26) the pressure at any point
... ...ssure at a depth z below the *common*
pressure at any point of the base is

...idth of one of the vertical sides; the
...d upon it $= \frac{1}{2} g\rho bh^2$, and the pressure of

$$\cdots \rho' z)\, bdz = gbh' \left(\rho h + \tfrac{1}{2}\rho' h'\right).$$

· is the sum of these two and is equal to

$$gb \left(\tfrac{1}{2}\rho h^2 + \rho hh' + \tfrac{1}{2}\rho' h'^2\right).$$

the fluid pressure on this side about its line
the surface

$$\int_0^h g\rho bz^2 dz + \int_h^{h+h'} g\,(\rho h + \rho' z)\, bzdz:$$

...tegrations, and dividing by the expression for
...sure investigated above, we obtain the depth of
...ssure.

To find the resultant vertical pressure on any ...ogeneous inelastic fluid at rest under the action of

a surface exposed to the action of a heavy fluid;
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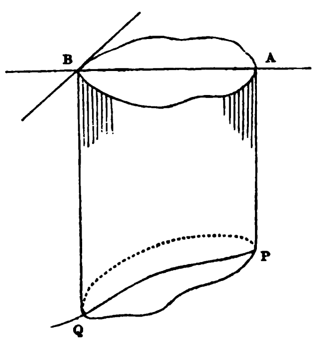


...id AQ is supported by
...al pressure of the fluid,
...e reaction of PQ; this
...solved vertically must be
...e weight of PQ, and con-
...e pressure on PQ is equal to the weight of AQ, and
...gh its centre of gravity.

43. Hence, in general, to determine the resultant fluid pressure on any surface, find the vertical pressure, and the resultant horizontal pressures in two directions at right angles to each other. These three forces may in some cases be compounded into a single force, the condition for which may be determined by the usual methods of Statics.

Ex. A hemisphere is filled with homogeneous fluid: required to find the resultant action on one of the four portions into which it is divided by two vertical planes through its centre at right angles to each other.

Taking the centre O as origin, the bounding horizontal radii as axes of x and y, and the vertical radius as the axis of z, the pressure parallel to x is equal to the pressure on the quadrant yOz, which is the projection, on a plane perpendicular to Ox, of the curved surface,

$$=g\rho\frac{\pi a^2}{4}\cdot\frac{4a}{3\pi}=\frac{1}{3}g\rho a^3,$$

and the co-ordinates of its point of action are

$$\left(o,\ \frac{3}{8}a,\ \frac{3}{16}\pi a\right),$$

Art. 39; similarly, the pressure parallel to $Oy=\frac{1}{3}g\rho a^3$, and acts through the point,

$$\left(\frac{3}{8}a,\quad o,\quad \frac{3}{16}\pi a\right).$$

The resultant vertical pressure = the weight of the fluid $=\frac{1}{6}g\rho\pi a^3$, and acts in the direction of the line $x=\frac{3}{8}a=y$.

The directions of the three forces all pass through the point

$$\left(\frac{3}{8}a,\ \frac{3}{8}a,\ \frac{3}{16}\pi a\right),$$

and they are therefore equivalent to a single force

$$\frac{1}{6}g\rho a^3\sqrt{(\pi^2+8)}\ \text{ in the line}$$

$$x-\frac{3}{8}a=y-\frac{3}{8}a=\frac{2}{\pi}\left(z-\frac{3}{16}\pi a\right),$$

$$\text{or } x = y = \frac{2}{\pi} z,$$

a straight line through the centre, as must obviously be the case, since all the fluid pressures are normal to the surface. The point in which it meets the surface of the hemisphere may be called 'the centre of pressure.'

44. PROP. *To find the resultant pressure on the surface of a solid either wholly or partially immersed in a heavy inelastic fluid.*

Suppose the solid removed, and the space it occupied filled with fluid of the same kind, and conceive this fluid solidified; the resultant pressure upon it will be the same as upon the orignal solid. But the solidified fluid is at rest under the action of its own weight, and the pressure of the fluid surrounding it: the resultant pressure is therefore equal to the weight of the fluid displaced, and acts in a vertical line through its centre of gravity*.

The same reasoning evidently shews that the resultant pressure of an elastic fluid on any solid is equal to the weight of the elastic fluid displaced by the solid.

45. PROP. *To find the resultant pressure on any surface of a fluid at rest under the action of any given forces.*

Let p be the pressure, determined as in Chapter II., at any point (x, y, z) of a surface, $u = 0$, exposed to the action of the fluid. Then if

$$\frac{1}{P^2} = \left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2,$$

$$P\frac{du}{dx}, \quad P\frac{du}{dy}, \quad P\frac{du}{dz},$$

are the direction-cosines of the normal at the point (x, y, z).

---

* This result may also be obtained by means of Arts. 41 and 42, as follows: Draw parallel horizontal lines touching the surface, and forming a cylinder which encloses it; the curve of contact divides the surface into two parts, on which the resultant horizontal pressures, parallel to the axis of the cylinder, are by Art. 42 equal and opposite; the horizontal pressures on the solid therefore balance each other and the resultant is wholly vertical. To determine the amount of the resultant vertical pressure, draw parallel vertical lines touching the surface, and dividing it into two portions on one of which the resultant vertical pressure acts upwards, and on the other downwards; the difference of the two, by Art. 41, is evidently the weight of the fluid displaced by the solid.

Let δS be an element of the surface about the same point. The pressures on this element, parallel to the axes, are

$$pP\frac{du}{dx}\,\delta S, \qquad pP\frac{du}{dy}\,\delta S, \qquad pP\frac{du}{dz}\,\delta S;$$

$\therefore$ if X, Y, Z, and L, M, N, be the resultant pressures parallel to the axes, and the resultant couples, respectively,

$$X = \iint pP\frac{du}{dx}\,\delta S, \quad Y = \iint pP\frac{du}{dy}\,\delta S, \quad Z = \iint pP\frac{du}{dz}\,\delta S,$$

$$L = \iint pP\,\delta S\left(y\frac{du}{dz} - z\frac{du}{dy}\right),$$

$$M = \iint pP\delta S\left(z\frac{du}{dx} - x\frac{du}{dz}\right),$$

$$N = \iint pP\delta S\left(x\frac{du}{dy} - y\frac{du}{dx}\right);$$

the integrations being made to include the whole of the surface under consideration.

These resultants are equivalent to a single force if

$$XL + YM + ZN = 0.$$

46. The surface may be divided into elements in three different ways by planes parallel to the co-ordinate planes.

Thus, $\delta x\delta y$ = projection of δS on $xy = P\dfrac{du}{dz}\,\delta S$;

and $\therefore Z = \iint p\,dxdy$; and similarly, $X = \iint p\,dydz$, and $Y = \iint p\,dzdx$,

$$L = \iint p\,(ydxdy - zdxdz),$$
$$= \iint p\,(ydy - zdz)\,dx,$$
$$M = \iint p\,(zdz - xdx)\,dy,$$
$$N = \iint p\,(xdx - ydy)\,dz.$$

47. If the fluid be at rest under the action of gravity only, and the axis of z be vertical, p is a function of z, $\phi(z)$ suppose, and therefore,

$$X = \iint \phi(z)\,dydz,$$

which is evidently the expression for the pressure, parallel to x, upon the projection of the given surface on the plane yz; and similarly Y is equal to the pressure upon the projection on xz.

Again, if the fluid be incompressible and acted upon by gravity only, $pdxdy$ is equal to the weight of the portion of fluid contained between δS and its projection on the surface;

$\therefore$ Z or $\iint pdxdy$, is the weight of the superincumbent fluid.

These results accord with those previously obtained, Arts. 41 and 42.

EXAMPLES.

1. Find the centres of pressure of a parallelogram with one side in the surface, and of a triangle with one side in the surface.

2. Water is poured into a hollow sphere, determine the depth of the water when the resultant pressure is half the total normal pressure.

3. A conical wine-glass is filled with water and placed in an inverted position upon a table; if the whole pressure of the water on the glass be double its resultant pressure, find the vertical angle of the cone.

4. A hollow paraboloidal vessel, open at the top, is inverted and placed on a horizontal table; fluid being poured in through a hole at the vertex, find its height when it begins to escape, and the condition that this may be possible.

5. Find the centre of pressure of a square lamina having one angular point in the surface of a fluid; and supposing it to be moved about the angular point in its own plane, which is fixed, and to be always totally immersed, find the locus of its centre of pressure.

6. Find the centre of pressure of an elliptic lamina just immersed in a heavy fluid; and supposing it turned round in the same vertical plane, so as to be always just immersed, find the locus with respect to its axes of the centre of pressure.

7. A cubical box, filled with water, has a close-fitting heavy lid fixed by smooth hinges to one edge; compare the tangents

of the angles through which the box must be tilted about the several edges of its base, in order that the water may just begin to escape.

8. Find the centre of pressure of a semi-ellipse (axes $2a$ and a) which is bounded by a diameter inclined at the angle $\frac{\pi}{6}$ to its major axis, its plane being vertical, and the diameter in the surface.

9. If a plane area immersed in a fluid revolve about any axis in its own plane, prove that the centre of pressure describes a straight line in the plane.

10. The initial radius lying in the surface, find the centre of pressure of the loop of the curve $r = a \cos 2\theta$, which is completely immersed. Also find the centre of pressure of one of the half loops immersed.

11. A plane area is wholly immersed in a fluid in a position not horizontal; if the area be turned about its centre of gravity in its own plane, shew that the locus of the centre of pressure of the area will be an ellipse.

12. A square board immersed in fluid is moveable about a horizontal side as a fixed axis; shew that the locus of the centre of pressure is defined by the equation

$$r = \frac{a}{3} \cdot \frac{3c - 2a \sin \theta}{2c - a \sin \theta},$$

where a is the side of the square, c the depth of the fixed side below the surface, and r the distance of any point in the locus from the fixed axis; the prime radius being horizontal.

13. A hollow cube is very nearly filled with fluid, and is made to rotate uniformly about a vertical edge; find the pressure upon, and centres of pressure of, its several sides.

14. A closed cylinder, very nearly filled with fluid, rotates uniformly about a generating line, which is vertical; find the resultant pressure on its curved surface.

Determine also the point of action of the pressure on its upper end.

15. A cube is filled with a fluid the density of which varies as the depth. The diagonal which passes through the highest corner of the cube makes angles α, β, γ, with the vertical. Find the resultant pressure on one of the upper faces of the cube.

16. A cone is filled with fluid, and fitted with a heavy lid, moveable about a hinge; it is then made to revolve uniformly about the generating line through the hinge, which is placed vertical; find the greatest angular velocity consistent with no escape of the fluid.

17. A cube whose edge is $2a$, and whose faces are horizontal and vertical, is surrounded by a mass of heavy fluid, the volume of which is $8a^3\{\pi\sqrt{6}-1\}$; the fluid is acted on by a force tending to the centre of the cube, and varying as the distance, the force at the distance a being g; find the form of the free surface and the pressure at any point: also if one of the vertical faces of the cube be moveable about a horizontal line in its own plane, shew that the face will be at rest, if this line be at a distance $\frac{4}{5}a$ from the lowest edge of that face.

18. A hollow cone open at the top is filled with fluid; find the resultant pressure on the portion of its surface cut off, on one side, by two planes through its axis inclined at a given angle to each other; also determine the line of action of the resultant pressure, and shew that, if the vertical angle be a right angle, it will pass through the centre of the top of the cone.

19. A quantity of incompressible fluid acted upon by a central force varying as the distance is contained between two parallel planes; if A, B, be the areas of the planes in contact with the fluid, shew that the pressures upon them are in the ratio $A^2 : B^2$.

20. A hollow sphere is full of fluid, the density of which varies as the $(\text{depth})^n$; shew that the whole pressure on the surface of the sphere : the resultant pressure :: $n+3 : n+1$.

21. Water in a vessel completely full is made to rotate uniformly about a horizontal axis; find the surfaces of equal pressure.

22. A vessel in the form of a surface of revolution has the following property; if it be placed with its axis vertical, and any quantity of fluid be poured into it, the ratio of the total normal pressure to the resultant vertical pressure varies as the depth of the fluid poured in. Shew that the equation to the generating curve is

$$cs = xy.$$

CHAPTER IV.

THE EQUILIBRIUM AND THE OSCILLATIONS OF FLOATING BODIES.

48. PROP. *To find the conditions of equilibrium of a floating body.*

We shall suppose that the fluid is at rest under the action of gravity only, and that, under the action of the same force, the body is floating freely in the fluid. The only forces then which act on the body are its weight, and the pressure of the surrounding fluid, and in order that equilibrium may exist, the resultant fluid pressure must be equal to, and act in an opposite direction to, the weight of the body. Now we have shewn, Art. (43), that the resultant pressure of a heavy fluid on the surface of a solid, either wholly or partially immersed, is equal to the weight of the fluid displaced, and acts in a vertical line through its centre of gravity.

Hence it follows that the weight of the body must be equal to the weight of the fluid displaced, and that the centres of gravity of the body, and of the fluid displaced, must lie in the same vertical line.

These conditions are necessary and sufficient conditions of equilibrium, whatever be the nature of the fluid in which the body is floating. If the fluid be heterogeneous, the displaced fluid must be looked upon as following the same law of density as the surrounding fluid; in other words, it must consist of strata of the same kind, and continuous with, the horizontal strata of uniform density, in which the particles of the surrounding fluid are necessarily arranged.

If for instance a solid body float in water, partially immersed, its weight will be equal to the weight of the water displaced,

together with the weight of the air displaced; and if the air be removed, or its pressure diminished by a diminution of its density or temperature, Art. (19), the solid will *sink* in the water through a space depending upon its own weight, and upon the densities of air and water. This may be further explained by observing that the pressure of the air on the water is greater than at any point above it, and that this *surface* pressure of the air is transmitted by the water to the immersed portion of the floating body, and consequently the *upward* pressure of the air upon it is greater than the *downward* pressure.

49. We now proceed to illustrate the application of the above conditions, by the discussion of some particular cases.

Ex. 1. A portion of a solid paraboloid, of given height, floats with its axis vertical and vertex downwards in a homogeneous fluid, required to find its position of equilibrium.

Taking $4a$ as the latus rectum of the generating parabola, h its height, and x the depth of its vertex, the volumes of the whole solid and of the portion immersed are respectively $2\pi ah^2$ and $2\pi ax^2$; and if ρ, σ, be the densities of the solid and fluid, one condition of equilibrium is

$$\rho \cdot 2\pi ah^2 = \sigma \cdot 2\pi ax^2;$$

$$\therefore\ x = \sqrt{\frac{\rho}{\sigma}}\,h,$$

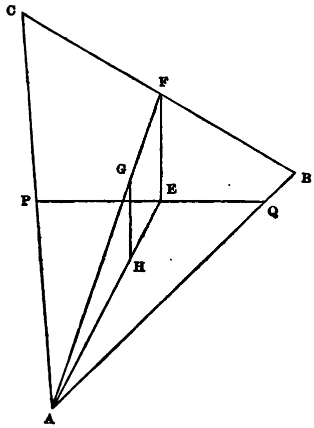
which determines the portion immersed, the other condition being obviously satisfied.

Ex. 2. A triangular prism floats in a fluid with its edges horizontal, to find its positions of equilibrium.

Let the figure be a section of the prism by a vertical plane through its centre of gravity.

PQ is the line of floatation and H the centre of gravity of the fluid displaced. When there is equilibrium the area APQ is to ABC in the ratio of the density of the prism to the density of the fluid, and therefore for all possible positions of PQ the

area APQ is constant; hence PQ always touches, at its middle point, a hyperbola of which AB, AC, are the asymptotes.



Also HG must be perpendicular to PQ, and therefore since

$$AH : HE = AG : GF,$$

FE must be perpendicular to PQ, that is, FE is the normal at E to the hyperbola. The Problem is therefore reduced to that of drawing normals from F to the curve*.

Let $xy = c^2$ be the equation to the curve referred to AB, AC as axes, and let

$$\angle BAC = \theta,\ AB = 2a,\ AC = 2b \ldots\ldots\ldots\ldots (a).$$

Let x, y, be the co-ordinates of E; the co-ordinates of F are a, b, and the equation to the normal at E is

$$\eta - y = \frac{y \cos\theta - x}{x \cos\theta - y} (\xi - x).$$

And if this pass through F, the co-ordinates of which are a, b,

$$(b - y)(x \cos\theta - y) = (a - x)(y \cos\theta - x),$$

$$\text{or}\quad x^2 - (a + b\cos\theta)\, x = y^2 - (a\cos\theta + b)\, y \ldots\ldots (\beta).$$

* *Senate-House Problems*, 1852.

The equations (α) and (β) determine all the points of the hyperbola, the tangents at which can be lines of floatation.

Also (β) is the equation to an equilateral hyperbola, referred to conjugate diameters parallel to AB, AC; the points of intersection of the two hyperbolas are therefore the positions of E.

To find x, we have

$$x^4 - (a + b\cos\theta) \,.\, x^3 + (a\cos\theta + b)\, c^2 x - c^4 = 0,$$

an equation which has only one negative root, and one or three positive roots, and there may be therefore three positions of equilibrium or only one.

If the densities of the fluid and the prism be ρ and σ, we have, since the area PAQ

$$= \tfrac{1}{2} AP \,.\, AQ \sin\theta = 2xy\sin\theta = 2c^2 \sin\theta,$$

$$2\rho c^2 \sin\theta = 2\sigma ab \sin\theta,$$

$$\text{or} \quad \rho c^2 = \sigma ab,$$

from which c is determined.

Suppose the prism to be isosceles, then putting $a = b$, the equation for x becomes

$$x^4 - c^4 - a(1 + \cos\theta)(x^3 - c^2 x) = 0;$$

from which we obtain $x = c$, which gives $y = c$, and makes BC horizontal, an obvious position of equilibrium, and also

$$x = \frac{a}{2}(1 + \cos\theta) \pm \left\{\frac{a^2}{4}(1 + \cos\theta)^2 - c^2\right\}^{\frac{1}{2}}$$

$$= a\cos^2\frac{\theta}{2} \pm \left(a^2 \cos^4\frac{\theta}{2} - c^2\right)^{\frac{1}{2}};$$

the isosceles prism will therefore have only one position of equilibrium, unless

$$a\cos^2\frac{\theta}{2} > c;$$

and since $\rho c^2 = \sigma a^2$ this is equivalent to

$$\cos^2\frac{\theta}{2} > \sqrt{\frac{\sigma}{\rho}}^*.$$

Ex. 3. Determine the position of equilibrium of a balloon of given size and weight, neglecting the variations of temperature at different heights in the atmosphere.

If the temperature be constant, the pressure of the air at a height $z = \Pi\epsilon^{-\frac{gz}{k}}$, and its density $= \frac{\Pi}{k}\epsilon^{-\frac{gz}{k}}$, Π being the atmospheric pressure at the level from which the height is measured.

The air displaced consists of a series of strata of variable density, and if z be the height of the lowest point of the balloon, x the distance from that point of any horizontal section (X) of the balloon, and h its height, the weight of a stratum of the air displaced is

$$\frac{\Pi g}{k}\epsilon^{-\frac{g(z+x)}{k}}X\delta x,$$

and the whole weight of air displaced

$$= \int_0^h \frac{\Pi g}{k}\epsilon^{-\frac{g(z+x)}{k}}Xdx = \frac{\Pi g}{k}\epsilon^{-\frac{gz}{k}}\int_0^h \epsilon^{-\frac{gx}{k}}Xdx.$$

The form of the balloon being given, X is a known function of x, and if W be the weight of the balloon and of the gas it contains, the height z will be determined by equating W to the expression we have obtained for the weight of the air displaced.

Ex. 4. A homogeneous solid floats, wholly immersed, in a fluid of which the density varies as the depth; to find the depth of its centre of gravity.

Let a, c, be the depths of the highest and lowest points of

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* *Note on Ex.* 2. It is evident that, if any solid float so that the immersed portion is a triangular prism, the construction employed in Ex. 2 will determine its positions of equilibrium; for if G be its centre of gravity, and AG be joined and produced to F so that $AG = 2GF$, F is a fixed point through which the normal at E must always pass.

the solid, Z the area of a horizontal section of the solid at a depth z, and μz the density;

$$\text{the weight of the fluid displaced} = \int_a^c g\mu z Z dz.$$

Let $\bar{z}$ be the depth of the centre of gravity of the solid and V its volume, then

$$V\bar{z} = \int_a^c Zz dz;$$

therefore the weight of displaced fluid $= g\mu\bar{z}V$, and if ρ be the density of the solid, its weight $= g\rho V$; hence $\rho = \mu\bar{z}$, or the solid floats in such a position that the density of the fluid at the depth of its centre of gravity is equal to its own density.

50. If a solid float *under constraint*, the conditions of equilibrium depend on the nature of the constraining circumstances, but in any case the resultant of the constraining forces must act in a vertical direction, since the other forces, the weight of the body, and the fluid pressure, are vertical.

If for instance one point of a solid be fixed, the condition of equilibrium is that the weight of the body and the weight of the fluid displaced should have equal moments about the fixed point; this condition being satisfied, the solid will be at rest, and the strain on the fixed point will be the difference of the two weights.

As an additional illustration, consider the case of a solid floating in water and supported by a string fastened to a point above the surface: in the position of equilibrium the string will be vertical, and the tension of the string, together with the resultant fluid pressure, which is equal to the weight of the displaced fluid, will counterbalance the weight of the body; the tension is therefore equal to the difference of the weights, and the weights are inversely in the ratio of the distances of their lines of action from the line of the string, these three lines being in the same vertical plane.

51. Prop. *A solid of revolution floats in a fluid which rotates uniformly about a vertical axis, the axis of the solid coinciding with the axis of rotation; required to find the condition of equilibrium.*

In a mass of rotating fluid, suppose a surface of revolution described, having its axis coincident with the axis of rotation, and let the fluid within this surface be made solid. No immediate change of motion will be produced, and since the rotation is about a principal axis, and the fluid pressures on the solidified fluid are normal to its surface, no subsequent change will take place, and the solidified fluid will continue to rotate as before. The resultant of the fluid pressures upon this solid is therefore equal to its weight, and the same pressures being exerted on the surface of any solid occupying the same space, it follows that any such solid will be in equilibrium, if its weight be equal to the weight of the fluid it displaces.

It will be seen moreover that it is quite indifferent whether the solid rotate with the fluid, or with a different angular velocity, or be at rest.

52. Ex. A cylinder floats in rotating fluid; to find the depth to which it is immersed.

If ω be the angular velocity, the equation to the generating parabola of the free surface, taking its vertex as the origin, is $\omega^2y^2 = 2gx$, and if z be the depth of the base of the cylinder below the circle of floatation, that is, the circle in which the free surface intersects the surface of the cylinder, and c the radius of the cylinder, the volume of the displaced fluid is the difference between the volume of a height z of the cylinder, and the volume of a height $\dfrac{\omega^2c^2}{2g}$ of the paraboloid.

Hence, if σ be the density of the cylinder and ρ of the fluid,

$$\sigma\pi c^2h = \rho\left(\pi c^2z - \frac{\pi\omega^2c^4}{4g}\right),$$

$$\text{and } z = \frac{\sigma}{\rho}h + \frac{\omega^2c^2}{4g}.$$

The stability of the equilibrium of floating bodies.

53. If a floating body be slightly displaced, it will in general either tend to return to its original position, or will recede farther from that position; in the former case the equilibrium is said to be *stable*, and in the latter *unstable*, for that particular direction of displacement.

Consider first a small vertical displacement: it is clear, if the body be floating partially immersed in homogeneous fluid, or if it be immersed, either wholly or partially, in a heterogeneous fluid of which the density increases with the depth, that a depression will increase the weight of the fluid displaced, and on the contrary an elevation will diminish it; in either case the tendency of the fluid pressure is to restore the body to its position of rest, and the equilibrium is *stable* with regard to vertical displacements. This, it will be observed, is only shewn to be true of *rigid* bodies; if the increased pressure, caused by depression, have the effect of compressing any portion of the floating body, the equilibrium is not necessarily *stable*, and in fact it may be *unstable*.

54. An arbitrary displacement will in general involve both vertical and angular changes in the position of the body; if however the displacement be small, as we have supposed to be the case, the effects of the two changes of position can be treated independently; and we proceed to consider the effect of a small angular displacement, on the supposition that the weight of fluid displaced remains unchanged, and consequently that the fluid pressure has no tendency to raise or depress the centre of gravity of the body.

For the proposed investigation, the following geometrical proposition will be found important.

If a solid be cut by a plane, and this plane be made to turn through a very small angle about a straight line in itself, the volume cut off will remain the same, provided the straight line pass through the centre of gravity of the area of the plane section.

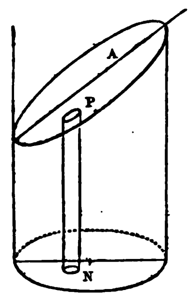
To prove this, consider a right cylinder of any kind cut by a plane making with its base an angle θ.

Let $\bar{z}$ be the distance from the base of the centre of gravity of the section A, δA an element of the area of the section and V the volume between the planes. Then

$$\bar{z} = \frac{\Sigma\,(\delta A \,.\, PN)}{A};$$

$$\therefore\ A\cos\theta\bar{z} = \Sigma(\delta A\cos\theta\,.\,PN) = V,$$

$$\text{or } V = \bar{z}\,(\text{area of base}).$$



Now the centre of gravity of the area A is also the centre of gravity of all sections made by planes passing through it, as may be seen by projecting the sections on the base of the cylinder; it follows therefore, that, $\bar{z}$ being the same for all such sections the volumes cut off are the same.

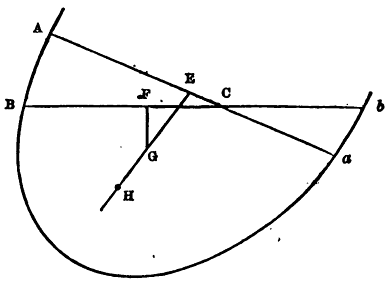
In the case of any solid, if the cutting plane be turned through a very small angle about the centre of gravity of its section, the surface near the curves of section may be considered, without sensible error, cylindrical, and the above proposition is therefore established.

In other words, the difference between the volume lost and the volume gained by the change in the position of the cutting plane will be indefinitely small compared with either.

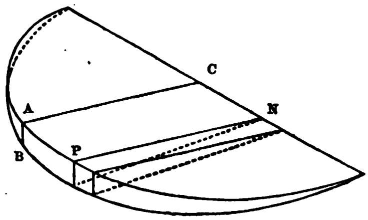
55. *A solid, floating at rest in a homogeneous fluid, is made to turn through a very small angle in a given vertical plane; to determine whether the fluid pressure will tend to restore it to its original position or not.*

Take the case in which the body is symmetrical with respect to the vertical plane through its centre of gravity in the direction of which it is displaced; and suppose the volume of fluid displaced to be unchanged.

Let AEC be the original plane of floatation and BC the direction of the fluid surface after displacement, G the centre of gravity of the solid, H of the fluid originally displaced, and V the volume of the fluid displaced.



In the 2nd figure CN is the line of intersection of the two planes ACa, BCb, which is perpendicular to the plane ACB, in the first figure.



The resultant fluid pressure is the weight of Bab acting upwards, and is therefore equivalent to the weight of ABa acting upwards, of the wedge aCb acting upwards, and of the wedge ACB acting downwards.

The volume of an element PN of the wedge,

$$= \tfrac{1}{2} PN^2 \, \delta\theta \, \delta x,$$

when $x = CN$, and the distance from CN of its centre of gravity is $\tfrac{2}{3} PN$; hence, taking the quantity $g\rho$ to be unity, the moment of the fluid pressures about a horizontal axis through G parallel to CN

$$= V . HG \sin \delta\theta - \Sigma \left(\tfrac{1}{3} PN^3 \, \delta\theta \, \delta x\right),$$

the summation extending through both wedges,

$$\text{and } \therefore \ = (V . HG - Ak^2) \, \delta\theta,$$

where A is the area Aa, and k its radius of gyration relative to the line CN.

If this moment is negative the solid tends to return to its original position, i. e. the equilibrium is stable, if

$$HG < \frac{Ak^2}{V},$$

and conversely, is unstable, if

$$HG > \frac{Ak^2}{V}.$$

If M be the point in HG through which the resultant vertical pressure of the fluid acts, in other words, if the vertical line through the centre of gravity of the fluid displaced meet HG in M, the moment is

$$V . GM \sin \delta\theta,$$

$$\text{or } V(HG - HM) \, \delta\theta;$$

$$\therefore \ HM = \frac{Ak^2}{V},$$

and the equilibrium is stable or unstable according as $HM >$ or $< HG$.

The point M is called the *metacentre*.

If $HG = \dfrac{Ak^2}{V}$, that is, if M and G coincide, the equilibrium is said to be neutral.

56. If the plane of displacement do not divide the body symmetrically, the expression

$$(V.HG - Ak^2)\,\delta\theta,$$

will still represent the moment of the fluid pressures, but the line of action of the resultant fluid pressure will not necessarily lie in the plane ABa.

Let $\bar{x}$ be the distance measured in the direction CN, 2nd figure, of the vertical through the centre of gravity (H') of the solid Bab, then,

$$V\bar{x} = \delta\theta \,.\int_{-x'}^{x''} \frac{1}{2}\, xQM^2dx - \delta\theta \,.\int_{-x'}^{x''} \frac{1}{2}\, xPN^2dx,$$

where QM refers to the wedge aCb, and x', x'' are the extreme values of x; and if y, an ordinate corresponding to either part of the plane of floatation, be considered positive when measured in the direction Ca, this may be written

$$V\bar{x} = \delta\theta \int_{-x'}^{x''} \frac{1}{2}\, xy^2dx.$$

If the projection of the vertical through H' on the plane ABa meet HG in M, the moment of the fluid pressures about G will still be represented by $V.\,GM\delta\theta$, and therefore as in the previous case $V.HM = k^2A$, and if rotation in the direction of the plane ABa only be allowed, the position of the point M defines the stability of the equilibrium.

57. It must be observed that the above investigation is essentially statical; it is simply an inquiry into the direction in which the moment of the fluid pressure about a certain horizontal axis through G is acting in the position of displacement contemplated.

Considered dynamically, if the horizontal axis through G be not a principal axis, the forces introduced by displacement will cause accelerations about other axes through G, and will consequently produce rotations about varying axes.

Moreover a rotation about G would, except in the case in which F and C, Art. 55, are coincident, cause a change in the quantity of fluid displaced, and vertical oscillations would therefore ensue.

58. Ex. 1. A solid cone floating with its axis vertical and vertex downwards.

Let h be the length of the axis,

z the portion of the axis immersed,

2α the vertical angle of the cone.

Then $$Ak^2 = \pi z^2 \tan^2\alpha \,.\, \frac{z^2\tan^2\alpha}{4},$$

$$\text{and } V = \frac{1}{3}\pi z^3 \tan^2\alpha;$$

$$\therefore\ HM = \frac{3}{4} z \tan^2\alpha;$$

$$\text{also } HG = \frac{3}{4} h - \frac{3}{4} z,$$

and therefore the equilibrium is stable or unstable, according as

$$z \tan^2\alpha > \text{ or } < h - z,$$

$$\text{or} \qquad z > \text{ or } < h\cos^2\alpha.$$

But if ρ, σ, be the densities of the fluid and cone,

$$\left(\frac{z}{h}\right)^3 = \frac{\sigma}{\rho};$$

therefore the equilibrium is stable or unstable as

$$\frac{\sigma}{\rho} > \text{ or } < (\cos\alpha)^6.$$

Ex. 2. An isosceles triangular prism floating with its base not immersed, and its edges horizontal.

Referring to Art. 49, consider first the position of equilibrium in which the base is inclined to the horizon.

In this case, if $AQ = 2y$ and $AP = 2x$, x and y are given by the equations

$$x + y = 2a\cos^2\frac{\theta}{2},$$

$$xy = c^2.$$

The co-ordinates of G and H referred to AB, AC as axes are respectively,

$$\frac{2}{3}a, \quad \frac{2}{3}a, \quad \text{and } \frac{2}{3}x, \quad \frac{2}{3}y,$$

$$\therefore\ HG^2 = \frac{4}{9}\{(a-x)^2 + (a-y)^2 - 2(a-x)(a-y)\cos\theta\}$$

$$= \frac{4}{9}\{x^2 + y^2 + 2xy\cos\theta - 2a(1+\cos\theta)(x+y) + 2a^2(1+\cos\theta)\},$$

from which, by means of the above equations, we obtain

$$HG = \frac{4}{3}\sin\frac{\theta}{2}\left(a^2\cos^2\frac{\theta}{2} - c^2\right)^{\frac{1}{2}}.$$

The area $PAQ = 2c^2\sin\theta$, and if M be the metacentre, and l the length of the prism,

$$2lc^2\sin\theta \,.\, HM = \frac{PQ^2}{12}\,.\,PQ\,.\,l,$$

$$HM = \frac{PQ^3}{24c^2\sin\theta}.$$

But $\quad PQ^2 = 4(x^2 + y^2 - 2xy\cos\theta)$

$$= 16\cos^2\frac{\theta}{2}\left(a^2\cos^2\frac{\theta}{2} - c^2\right);$$

$$\therefore\ HM = \frac{4}{3}\,\frac{\cos^2\dfrac{\theta}{2}}{c^2\sin\dfrac{\theta}{2}}\left(a^2\cos^2\frac{\theta}{2} - c^2\right)^{\frac{3}{2}},$$

$$\text{and } HM > HG, \text{ if } c^2 \sin^2 \frac{\theta}{2} < \cos^2 \frac{\theta}{2} \left(a^2 \cos^2 \frac{\theta}{2} - c^2\right),$$

$$\text{i. e. if } \cos^2 \frac{\theta}{2} > \frac{c}{a}.$$

Next, consider the case in which the base is horizontal, and PQ therefore parallel to AB.

The area $PAQ = 2c^2 \sin\theta$,

$$AP = AQ = 2c, \text{ and } PQ = 4c \sin\frac{\theta}{2}.$$

$$\text{Hence, } HM = \frac{4}{3} c \frac{\sin^2 \frac{\theta}{2}}{\cos\frac{\theta}{2}}, \text{ and } HG = \frac{4}{3}(a - c)\cos\frac{\theta}{2},$$

$$\text{and } HM > HG \text{ if } \cos^2\frac{\theta}{2} < \frac{c}{a}.$$

Now in the Art. 49, before referred to, we have shewn that there are three positions of equilibrium, or one only, according as

$$\cos^2\frac{\theta}{2} > \text{ or } < \frac{c}{a}.$$

Hence it follows, that when there are three positions of equilibrium, the intermediate one, in which AB is horizontal, is a position of unstable equilibrium, while in the other two positions the equilibrium is stable.

If there be only one position in which the prism will rest, its equilibrium is stable.

Finite Displacements.

59. A more general case for consideration would be that of a body displaced through a finite angle from its position of equilibrium, the subject of investigation being the direction of the moment of the fluid pressure.

We shall consider only the case of a body which is symmetrical with respect to the vertical plane of displacement passing through its centre of gravity (G).

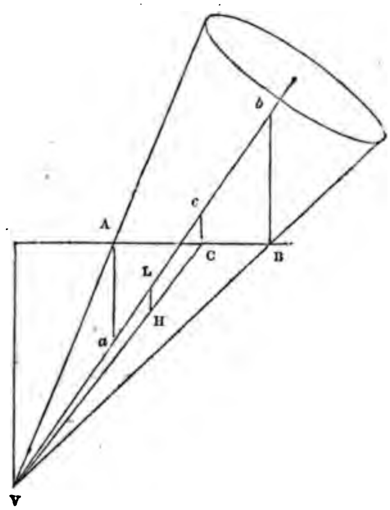
Let AG be the line through G in the body which is vertical in the position of equilibrium, H the centre of gravity of the fluid displaced, and L the point in which AG is intersected by the vertical through H.

Then it is clear that if $AL > AG$, the fluid pressure will *tend* to make AG vertical, and if $AL < AG$, will tend to turn AG farther from the vertical.

It is not to be inferred that if $AL > AG$, the body will when set free return to its original position and oscillate through it, or even that the original position is one of stable equilibrium, according to our previous definition of stability: it is a general law of mechanics that positions of stable and unstable equilibrium occur alternately, and the body may have been displaced from its original position *through* other positions of equilibrium.

60. As a particular example take the following.

A solid cone, floating with its axis vertical and vertex downwards, is turned through an angle θ in a vertical plane, the volume of fluid displaced remaining the same; to determine the direction of the moment of the fluid pressure.



Let AB be the major axis of the elliptic section made by the surface plane of the fluid, C its middle point, Aa, Bb, Cc,

lines at right angles to AB, and let the angle $AVB = 2\alpha$ and $VA = d$. Then

$$VAa = \theta - \alpha, \text{ and } VBb = \pi - \theta - \alpha.$$

$$Vc = \frac{1}{2}(Va + Vb) = \frac{1}{2}\cdot\left\{d\frac{\sin(\theta-\alpha)}{\sin\theta} + d\frac{\cos(\theta-\alpha)}{\cos(\theta+\alpha)}\frac{\sin(\theta+\alpha)}{\sin\theta}\right\}$$

$$= \frac{d\cos\theta}{\cos(\theta+\alpha)};$$

$$\therefore\ VL = \frac{3}{4}d\frac{\cos\theta}{\cos(\theta+\alpha)}.$$

The semi-minor axis of the ellipse AB is a mean proportional between the perpendiculars from A and B on the axis of the cone,

$$\therefore \text{ its area} = \pi\frac{1}{2}AB\,(VA\,.\,VB\,.\sin^2\alpha)^{\frac{1}{2}}$$

$$= \frac{\pi}{2}d^2\frac{\sin\alpha\sin 2\alpha}{\cos(\theta+\alpha)}\cdot\left\{\frac{\cos(\theta-\alpha)}{\cos(\theta+\alpha)}\right\}^{\frac{1}{2}};$$

therefore the volume of the fluid displaced

$$= \frac{1}{3}d\cos(\theta-\alpha)\,.\,(\text{area of ellipse})$$

$$= \frac{1}{3}\pi d^3\sin^2\alpha\cos\alpha\left\{\frac{\cos(\theta-\alpha)}{\cos(\theta+\alpha)}\right\}^{\frac{3}{2}}.$$

Hence, if ρ, σ, be the densities of the fluid and the cone, since the weight of the fluid displaced is equal to that of the cone, we have

$$\rho d^3\sin^2\alpha\cos\alpha\left\{\frac{\cos(\theta-\alpha)}{\cos(\theta+\alpha)}\right\}^{\frac{3}{2}} = \sigma h^3\tan^2\alpha,$$

$$\text{or } \left(\frac{d}{h}\right)^3 = \frac{\sigma}{\rho}\left\{\frac{\cos(\theta+\alpha)}{\cos(\theta-\alpha)}\right\}^{\frac{3}{2}}\frac{1}{\cos^3\alpha}.$$

And $VL > VG$ if

$$d\frac{\cos\theta}{\cos(\theta+\alpha)} > h,$$

or if
$$\sqrt[3]{\frac{\sigma}{\rho}} > \frac{\cos\alpha\cos(\theta+\alpha)}{\cos\theta}\cdot\left\{\frac{\cos(\theta-\alpha)}{\cos(\theta+\alpha)}\right\}^{\frac{1}{2}}.$$

Supposing θ indefinitely small, we obtain the condition of stability for an infinitesimal displacement,

$$\sqrt[3]{\frac{\sigma}{\rho}} > \cos^2\alpha\,; \text{ as before.}$$

Let the equilibrium of the cone be neutral, that is, let

$$\sigma = \rho\cos^6\alpha,$$

then, after a finite displacement, the action of the fluid will tend to restore the cone to its original position, if

$$\cos\alpha\,.\cos\theta > \sqrt{\{\cos(\theta+\alpha)\,.\cos(\theta-\alpha)\}},$$

a condition which is always true, α and θ being each less than a right angle.

In the case of neutral equilibrium of a cone, the equilibrium may therefore be characterised as stable for any finite displacement.

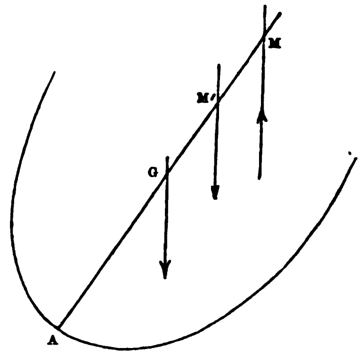
61. When fluid is contained in a vessel, which is slightly displaced from its original position, the preceding investigation enables us to determine the line of action of the resultant *downward* pressure of the fluid.

The problem in fact in this case, as in the previous case, is the following.

A given volume, the centre of gravity of which is H, is cut from a solid ABC by a plane, and the line CH is perpendicular to the plane; the same volume being cut off by a plane making a very small angle with the plane AB, to determine the position of the straight line perpendicular to the second plane, and passing through the centre of gravity of the volume cut off by it.

If the interior surface of the vessel is symmetrical with respect to the plane through H perpendicular to the line of intersection of the two planes, the line whose position is required will intersect CH in a point M, *the metacentre*, the position of which is determined by our previous results.

62. A hollow vessel containing fluid, floats in fluid; required to determine the nature of the equilibrium, supposing the body to be symmetrical with respect to the vertical plane of displacement through its centre of gravity, and that the centres of gravity of the body and of the fluid are in the same vertical line.



Let M be the metacentre for the displaced fluid, and M' for the contained fluid, W, W', the weights of the displaced and contained fluid.

Taking moments about G, the resultant fluid pressures will tend to restore equilibrium, or the reverse, according as

$$W \,.\, GM - W' \,.\, GM'$$

is positive or negative, i. e. as

$$\frac{W}{W'} > \text{ or } < \frac{GM'}{GM}.$$

63. Ex. A hollow cone containing water floats in water with its axis vertical.

Let $h =$ the length of the axis of the cone,

$h' =$ the length of the axis in the contained fluid,

$z =$ the length beneath the surface of the external fluid.

Taking 2α as the vertical angle of the cone, we have

$$HM = \frac{3}{4} z \tan^2 \alpha, \quad \text{Art. (58)};$$

But $$HG = \frac{2}{3} h - \frac{3}{4} z;$$

$$\therefore\ GM = \frac{3}{4} z \sec^2\alpha - \frac{2}{3} h.$$

Similarly $$GM' = \frac{3}{4} h' \sec^2\alpha - \frac{2}{3} h,$$

also $$\frac{W}{W'} = \frac{z^3}{h'^3};$$

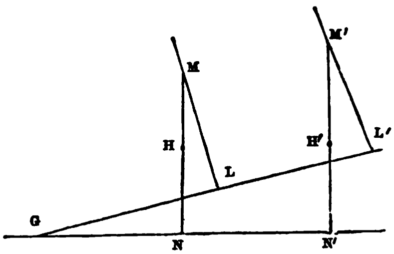
therefore the equilibrium is stable if

$$\left(\frac{z}{h'}\right)^3 > \frac{9h' \sec^2\alpha - 8h}{9z \sec^2\alpha - 8h},$$

z being given by the equation

$$W - W' = \frac{1}{3} g\rho\pi \tan^2\alpha\,(z^3 - h'^3) = \text{weight of cone.}$$

64. In the case in which the centres of gravity of the contained and of the fluid displaced are not in the same vertical, suppose the displacement to take place in direction of the vertical plane through the centres of gravity, and that the body is symmetrical with respect to that plane.



Let G be the centre of gravity of the body, H of the fluid displaced, H' of the contained fluid, and M, M', the metacentres.

Also let GNN' be horizontal in the position of equilibrium, and GLL' the horizontal line through G in the displaced position.

Then W, W', having the same meanings as before, and θ being the angle of displacement, the equilibrium is stable or unstable, as

$$W \,.\, GL > \text{or} < W' \,.\, GL',$$

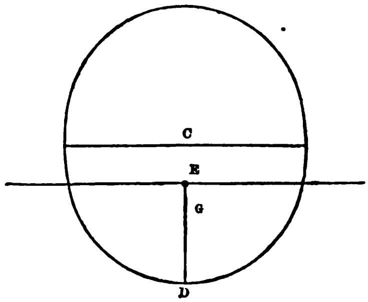
or $W(GN\cos\theta + MN\sin\theta) >$ or $< W'(GN'\cos\theta + M'N'\sin\theta)$,

i.e. since $W.GN = W'.GN'$,

as
$$\frac{W}{W'} > \text{ or } < \frac{M'N'}{MN}.$$

Oscillations of Floating Bodies.

65. We shall consider only the case in which the floating body is symmetrical with respect to a vertical plane through its



centre of gravity, and we shall suppose the initial displacement parallel to this plane.

It is evident that the subsequent motions of all points of the body will be parallel to this plane, and if the equilibrium be stable, that the motion will consist of small vertical and angular oscillations.

First let the vertical line through G and H (CED) pass through the centre of gravity of the plane of floatation. When this is the case we can consider the vertical and angular displacements independently of each other.

Suppose a small vertical displacement; then the portion CE of the body which is raised out of the fluid may be considered as a thin cylinder, and its weight will act downwards through C at the time t.

Let $CE = z$, then $EG = CG - z$.

The moving force downwards on the body = the weight of the body − the weight of the fluid displaced

$$= g\rho A . z,$$

if A be the area of the plane of floatation;

$$\therefore\ M\frac{d^2 . EG}{dt^2} = g\rho Az,$$

M being the mass of the body.

But Mg = the weight of fluid displaced

$= g\rho V$, V being the volume CD;

$$\therefore\ \frac{d^2z}{dt^2} + \frac{gA}{V}z = 0$$

is the equation which determines the motion.

The time of an oscillation is therefore $\pi\sqrt{\left(\frac{V}{gA}\right)}$.

Next suppose a small angular displacement (α) about C, then G is raised through a space which depends on α^2, and therefore may be neglected in comparison with quantities depending upon α, and if the body, supposed at rest, be then left to itself, it will (on the supposition that the equilibrium is stable) oscillate about a horizontal axis through G.

It would in fact come to the same thing if the initial displacement were about G, as the point C would move sensibly (that is, considering small quantities of the first order only,) in a horizontal direction, and the quantity of fluid displaced would, as before, remain unchanged.

The moment of the fluid pressure about G

$$= g\rho V . MG . \sin\theta,$$

and tends to diminish θ, the angle made by GH with the vertical at the time t.

But $$MG = \frac{k^2A}{V} - a, \text{ if } HG = a;$$

therefore, since the horizontal axis through G is a principal axis, we have

$$MK^2\frac{d^2\theta}{dt^2} = -g\rho\,(k^2A - aV)\,\theta,$$

neglecting higher powers of θ, where MK^2 is the moment of inertia of the body about the horizontal axis through G,

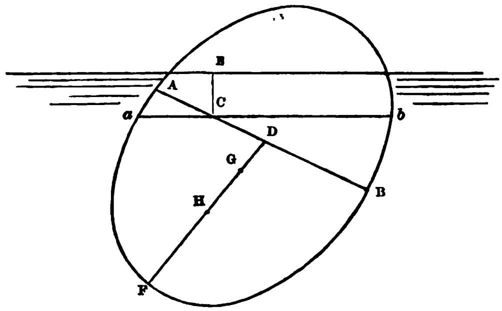
$$\text{or}\quad K^2\frac{d^2\theta}{dt^2}+g\left(\frac{k^2A}{V}-a\right)\theta=0.$$

An equation which, when $k^2A > aV$, that is, when M is above G, indicates small oscillations taking place in the time

$$\pi K\sqrt{\left\{\frac{V}{g\,(k^2A-aV)}\right\}}.$$

If G is below H the sign of a will of course be changed.

66. Secondly, when the line joining H and G does not pass through C, the two motions are not independent, but the law which defines these motions can be determined as follows.



Suppose the body to be slightly displaced in the vertical plane of symmetry, and then left to itself; and at the time t let θ be the angle made by HG with the vertical, and $z = CE$ the depth of C below the surface.

Let HG meet the plane of floatation in D,

$$HG=a,\quad CD=b,\quad DG=c,$$

and other symbols as before.

Then the depth of $G = z + b\sin\theta + c\cos\theta$

$$= z + b\theta + c,\ \text{to the order considered.}$$

The weight of the fluid displaced is the weight of a volume of fluid equal to

$$aFb + EC,\ \text{or}\ AFB + EC;$$

$$\text{this weight} = g\rho V + g\rho Az,$$

$$\text{and } \therefore \ M\frac{d^2}{dt^2}(z+c+b\theta) = Mg - (g\rho V + g\rho Az)$$

$$= -g\rho Az;$$

$$\text{or } \frac{d^2z}{dt^2} + b\frac{d^2\theta}{dt^2} = -g\frac{A}{V}.z \quad \text{..................... (I).}$$

Another equation is to be obtained from the consideration of the angular motion about the horizontal axis through G, which is a principal axis, perpendicular to the plane of displacement.

The moment of the fluid pressure about G may be divided into two parts, the one due to the portion aFb, and the other to the portion EC of the fluid displaced.

The former part of the fluid pressure $= g\rho V$ acting upwards through M the metacentre; the latter $= g\rho Az$, and may be considered to act through C the centre of gravity of the plane of floatation.

The moment, in the direction tending to diminish θ,

$$= g\rho V.\ GM \sin\theta - g\rho Az\,(b\cos\theta - c\sin\theta)$$

$$= g\rho\,(k^2A - aV)\,\theta - g\rho Az\,(b - c\theta)$$

$$= g\rho\,(k^2A - aV)\,\theta - g\rho Abz,$$

neglecting the product of z and θ;

$$\therefore \ MK^2\frac{d^2\theta}{dt^2} = -g\rho\,(k^2A - aV)\,\theta + g\rho Abz.$$

$$K^2\frac{d^2\theta}{dt^2} = -g\left(\frac{k^2A}{V} - a\right)\theta + g\frac{A}{V}.bz \quad \text{.......... (II).}$$

From the equations (I) and (II) we obtain

$$\frac{d^2z}{dt^2} + \frac{gA}{V}\left(1 + \frac{b^2}{K^2}\right)z - \frac{gb}{K^2}\left(\frac{k^2A}{V} - a\right)\theta = 0,$$

$$\frac{d^2\theta}{dt^2} - \frac{gAb}{VK^2}z + \frac{g}{K^2}\left(\frac{k^2A}{V} - a\right)\theta = 0,$$

which may be written

$$\frac{d^2z}{dt^2} + mz - bn\theta = 0,$$

$$\frac{d^2\theta}{dt^2} - \frac{pz}{b} + n\theta = 0.$$

To integrate these equations, multiply the second by λ, and add it to the first, then,

assuming $$\frac{\lambda n - bn}{mb - \lambda p} = \frac{\lambda}{b} \text{ .... (III)},$$

we have $$\frac{d^2}{dt^2}(z + \lambda\theta) + \left(m - \frac{\lambda p}{b}\right)(z + \lambda\theta) = 0,$$

and if λ_1 λ_2 be the roots of (III)

$$z + \lambda_1\theta = C_1 \cos\left\{\sqrt{m - \lambda_1\frac{p}{b}}\, t + \alpha_1\right\},$$

$$z + \lambda_2\theta = C_2 \cos\left\{\sqrt{m - \lambda_2\frac{p}{b}}\, t + \alpha_2\right\},$$

from which z and θ are completely determined.

The depth of G is given by an expression of the form

$$C + A\cos(\mu t + \alpha) + B\cos(\mu' t + \beta),$$

and its motion consists of two distinct oscillations, each following the pendulum laws, and compounded together in accordance with the principle of the coexistence of small oscillations*.

It may be observed that if two points be taken in the line AB, whose distances from C in the direction CD are λ_1, λ_2, then at the time t, the vertical depths of these points are $z + \lambda_1\theta$ and $z + \lambda_2\theta$, that is, are

$$C_1 \cos\left\{\sqrt{m + \lambda_1\frac{p}{b}}\, t + \alpha_1\right\}, \text{ and } C_2 \cos\left\{\sqrt{m + \lambda_2\frac{p}{b}}\, t + \alpha_2\right\},$$

and their vertical motions are therefore simple oscillations following the pendulum law. This remark is quoted by Duhamel (*Cours de Mécanique*, Art. 152) as due to M. Cauchy.

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* Poisson's *Cours de Mécanique*, Art. 618.

EXAMPLES.

1. A cylindrical block of wood is placed with its axis vertical in a cylindrical vessel whose base is plane, and water is then poured in to twice the height of the cylinder; find the pressure of the wood on the base of the vessel.

If the wood be displaced, and float, find the new pressure on that portion of the base with which it was previously in contact.

2. A cylinder floats between two fluids with its axis vertical, its height being equal to the depth of the upper fluid; compare the pressures on the two ends of the cylinder, the densities of the fluids and of the cylinder being given.

Is the equilibrium of the cylinder stable or unstable for a vertical displacement?

3. A cone, placed with its axis vertical and vertex downwards in a fluid, rests with half its axis immersed, and, when placed in another fluid, it rests with three-fourths immersed: in what proportion must these fluids be mixed, that it may rest in the mixture with two-thirds of the axis immersed?

4. A square board is placed in liquid of four times its density; shew that there are three different positions in which it will float with one given corner only below the surface of the fluid.

5. A porous body is weighed in air, and then attached to a sinker, and weighed in water; the weights in the two cases being 10 and 13 grs. respectively. It is then taken out of the water and weighed again, and in consequence of the water it has absorbed, its weight is increased to 12 gr. The weight of the sinker in water is 15 gr.: find the specific gravity of the solid part of the body; and the ratio that the cavities in the body bear to the whole volume.

6. A body is floating in a fluid; a hollow vessel is inverted over it and depressed: what effect will be produced in the position of the body, (1) with reference to the surface of the fluid

within the vessel, (2) with reference to the surface of the fluid outside?

7. The base of a vessel containing water is a horizontal plane, and a sphere of less density than water is kept totally immersed by a string fastened to the middle point of a circular disc, which lies in contact with the base. Find the greatest sphere of given density, and also the sphere of given size and least density, which will not raise the disc.

Examine also, in each case, the effect of increasing the density of the fluid, or of diminishing its depth.

8. A hollow hemispherical shell has a heavy particle fixed to its rim, and floats in a fluid with the particle just above the surface, and with the plane of the rim inclined at an angle of 45° to the surface; shew that the weight of the hemisphere : the weight of the fluid which it would contain

$$:: 4\sqrt{2}-5 : 6\sqrt{2}.$$

9. An elastic string has its upper extremity fixed, and the lower attached to the centre of the top of a heavy cylinder which rests partly immersed in water with its axis vertical; find the position of equilibrium.

10. A light body is imprisoned under water by a fixed plane surface; shew what will be the conditions of rest; and find the whole pressure on the plane, the dimensions of which are given.

11. From a cone of wood, of height h and specific gravity σ, a smaller cone of height h' is cut off, and this smaller cone is replaced by an equal and similar cone of metal of specific gravity σ'; shew that the cone thus formed will just float in a fluid of specific gravity s, if

$$h^3 : h'^3 :: \sigma' - s : s - \sigma.$$

12. Two equal slender rods AB, AC, moveable about a hinge at A, and connected by a string BC, rest with the angle A immersed in a given fluid; find the tension of the string.

13. A solid formed of two co-axial right cones, of the same vertical angle, connected at the vertices, is placed with one end

in contact with the horizontal base of a vessel: water is then poured into the vessel; shew that if the altitude of the upper cone be treble that of the lower, and the common density of the spindle four-sevenths that of the water, it will be upon the point of rising when the water reaches to the level of its upper end.

14. If to any floating body a second, weighing W in the fluid, be attached by a string, the position of the former when at rest will be the same as if it were free, and its weight were increased by W, and its centre of gravity moved towards the point where the string leaves the body by a distance depending upon W.

Find the positions of rest of an uniform plank of given dimensions, floating in water, and having a given weight attached by a string to the centre of one end. Shew whether they be positions of stable or unstable equilibrium.

15. A right circular cylinder floats in fluid with its axis in the surface, and is displaced through a small angle in the vertical plane passing through its axis; find the ratio between the length and radius of the cylinder when the equilibrium is neutral.

16. A cone, whose vertical angle is 60°, floats in water with its axis vertical; shew that its metacentre lies in the plane of floatation; and that its equilibrium will be stable provided its specific gravity $> \frac{27}{64}$.

17. An isosceles wedge floats with its base horizontal, and its edge immersed; shew that the equilibrium is stable for displacements in a plane perpendicular to the edge, if the ratio of the density of the wedge to that of the fluid is greater than the ratio $(\cos \alpha)^4 : 1$, 2α being the angle of the wedge.

18. An elliptical cylinder whose height is 5 inches, and the semi-axes of whose base are 4 inches and 3 inches, floats with its axis vertical in a fluid of double its specific gravity; in what directions is the equilibrium unstable?

19. A closed cylindrical vessel, quarter-filled with ice, is placed floating in water with its axis vertical; the weight of

the vessel is one-fourth of the weight of the water which it can contain; examine the nature of the equilibrium before and after the ice melts, neglecting the change of volume consequent on the change of temperature.

20. A sphere of given radius floats in equilibrium in a quantity of water contained in a cylindrical vessel, revolving uniformly about its axis which is vertical; the velocity of rotation is such that the centrifugal force at a distance from the axis equal to the radius of the sphere is equivalent to the force of gravity; prove that the whole pressure upon the sphere varies as the cube of the surface immersed.

21. A solid cone is divided into two parts by a plane through its axis, and the parts are connected by a hinge at the vertex; the system being placed in a fluid with its axis vertical and vertex downwards, shew that, if it float without separation of the parts, the length of the axis immersed is greater than $h \sin^2 \alpha$, h being the height of the cone, and 2α its vertical angle.

22. A hollow hemisphere moveable about a horizontal diameter is partly filled with fluid; shew that the time of a small oscillation is the same as if there were no fluid in it.

23. A solid, the lower portion of whose surface is spherical, floats in a heavy fluid; shew that the time of a small angular oscillation is the same in whatever fluid it floats.

24. A cone, the vertex of which is fixed at the bottom of a vessel containing fluid, is in equilibrium with its slant side vertical, and the lowest point of its base just touching the surface of the fluid. Compare the density of the cone with that of the fluid.

25. An ellipsoid floats in a fluid of twice its own specific gravity; find the times of small vertical and angular oscillations.

26. An uniform hemispherical shell, containing fluid, is placed on the vertex of a rough sphere of twice its diameter; shew that the equilibrium will be stable or unstable, as the

weight of the shell is greater or less than twice the weight of the fluid.

27. If a regular homogeneous tetrahedron, completely immersed, be placed in any position in a fluid whose density varies as the depth, shew that, when the resistance of the fluid is neglected, the tetrahedron will make vertical oscillations in the time $2\pi\sqrt{\left(\frac{h}{g}\right)}$, h being the depth of the centre of gravity of the tetrahedron in the position of equilibrium.

28. A solid of revolution floats in a fluid with its vertex downwards; determine its form when the position of the metacentre is independent of the density of the fluid.

29. A paraboloid of revolution floats with its axis vertical and vertex downwards in a fluid whose density varies as the depth; the equilibrium will be stable or unstable, according as $4c$ is less or greater than $3(m + a)$, where c is the length of the axis, a the length immersed, and m the latus rectum of the generating parabola.

30. An oblate spheroid is completely immersed in two fluids, the specific gravity of the lower being twice that of the upper fluid, and floats with its axis vertical, and its centre in the common surface of the fluids.

Supposing a small displacement to take place, 1st, in a vertical direction, 2ndly, about a horizontal line through its centre of gravity, shew that the times of the small oscillations will be respectively

$$\pi\sqrt{\left(\frac{2b}{g}\right)}, \text{ and } \pi\sqrt{\left(\frac{8}{5}\cdot\frac{b}{g}\frac{a^2+b^2}{a^2-b^2}\right)},$$

where a and b are the semi-axes of the generating ellipse.

31. A cylinder floats in a fluid contained in another cylinder; find how far the floating cylinder must be thrust down in order that on its return it may just rise out of the fluid.

32. Find a solid of revolution such that, when a segment of it is immersed in fluid, the distance between the centre of gravity

of the displaced fluid and the metacentre may be constant, whatever be the height of the segment.

33. A right cone rests in a vessel containing equal depths of two given fluids, with its vertex fastened to the bottom and its axis vertical. Find the condition for stable equilibrium, and, supposing it satisfied, find the time of a small oscillation.

34. A cylinder whose axis is vertical is floating in a fluid in which the density at any point varies as the n^{th} power of the depth; the cylinder is depressed till its upper end just coincides with the surface of the fluid, and on being let go it rises just out of the fluid; shew that, when the cylinder was floating, the depth immersed was to the height of the cylinder as 1 to $(n+2)^{\frac{1}{n+1}}$.

35. If a cylindrical shell without weight contain fluid and float in another fluid, shew that the equilibrium will be stable, unless the ratio of the density of the internal to the external fluid is less than unity, and greater than half the duplicate ratio of the radius of the cylinder to the depth of the internal fluid.

36. A cylindrical vessel is moveable about a horizontal axis passing through its centre of gravity, and is placed so as to have its axis vertical; if water be poured in, shew that the equilibrium is at first unstable; and find the condition which must be satisfied, in order that it may be possible to make the equilibrium stable by pouring in enough water.

37. A thin conical vessel of given weight is moveable about a diameter of its base which is horizontal, and is partly filled with a heavy fluid, shew that the equilibrium is always stable if the semi-vertical angle of the cone is $< 30^\circ$; and if it is greater than this, determine when the equilibrium is stable or unstable.

38. A cube half filled with fluid rests with the centre of its base upon the highest point of a rough sphere: determine the radius of the sphere so that the equilibrium may be stable.

39. A cone, whose height is h, oscillates with its axis vertical and vertex downwards in a fluid of uniform density. In the

highest position of the cone the vertex is in the surface, and in the lowest the cone is just immersed: shew that it would rest in a position of equilibrium with a length $\frac{h}{\sqrt[3]{4}}$ of its axis immersed.

40. A cube (the length of whose edge is $2a$) is floating in a fluid with its centre of gravity at a depth c below the surface; if it receive a small displacement so that two of its faces remain vertical, shew that the times of its small vertical and angular oscillations are

$$\pi\sqrt{\left(\frac{a+c}{g}\right)} \text{ and } 2\pi\sqrt{\left\{\frac{a^2(a+c)}{g(3c^2-a^2)}\right\}}, \text{ respectively.}$$

41. A closed cylinder is placed over a reservoir of water, and is filled with air under the atmospheric pressure, and the distance through which the cylinder sinks is observed: a quantity of gas, which under the atmospheric pressure would occupy three times the content of the cylinder, is introduced; determine the position of equilibrium and the motion of the cylinder.

42. A lamina of uniform thickness, in the form of an isosceles right-angled triangle, has one of the acute angles fixed below the surface of a fluid, and rests with the side which is not immersed horizontal. Prove that the time of a small oscillation in its own plane is $2\pi\sqrt{\left(\frac{a}{g}\right)}$, where a is the length of each of the sides of the triangle.

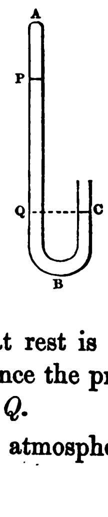
43. Water is contained in a vessel having a horizontal base, and a paraboloid whose specific gravity is four-ninths that of water, and the length of whose axis is to the latus rectum as nine to eight, is supported partly by the fluid and partly by the base on which the vertex rests; find the least depth of the fluid for which the equilibrium is stable.

CHAPTER V.

PRESSURE OF THE ATMOSPHERE.

67. If a glass tube, about three feet in length, having one end closed, be filled with mercury, and then inverted in a vessel of mercury so as to immerse its open end, it will be found that the mercury will descend in the tube, and rest with its upper surface at a height of about 29 inches above the surface of the mercury in the vessel: this experiment, first made by Torricelli, has suggested the use of the *Barometer*, for the purpose of measuring the atmospheric pressure.

The *Barometer*, in its simplest form, is a bent tube ABC, the branches of which AB, BC, are parallel and vertical, containing mercury: the end A is hermetically sealed, and there is no air in the branch AB.



It is found that the height of the surface P of the mercury above the surface C is about 29 inches, and, as there is no pressure on the surface P, it is clear that the pressure of the air on C is the force which sustains the column of mercury PQ.

We have shewn that the pressure of a fluid at rest is the same at all points of the same horizontal plane; hence the pressure at C is equal to the pressure of the mercury at Q.

Let σ be the density of mercury, and Π the atmospheric pressure at C, then

$$\Pi = g\sigma PQ,$$

and the height PQ measures the atmospheric pressure.

68. On account of its great density, mercury is the most convenient fluid which can be employed in the construction of

barometers, but the pressure of the air can be measured by using any kind of liquid. The density of mercury is about 13·568 times that of water, and therefore the height of the column of water in a water-barometer would be about $33\frac{3}{4}$ feet.

69. The density of mercury changes with the temperature, and σ must therefore be expressed as a function of the temperature.

Experiment shews that, for an increase of 1^0 centigrade, the expansion of mercury is $\frac{1}{5550}$th of its volume; hence if σ_t be the density at a temperature t^0, and σ_0 at a temperature 0^0,

$$\sigma_0 = \sigma_t\left(1 + \frac{t}{5550}\right) = \sigma_t(1 + \cdot 00018018t);$$

$$\therefore\ \sigma_t = \sigma_0(1 - \theta t) \text{ if } \theta = \cdot 00018018,$$

and $$\Pi = g\sigma_0(1 - \theta t)\,PQ.$$

70. By means of the formula, $\Pi = g\sigma_0(1 - \theta t)\,h$, the atmospheric pressure at any place can be calculated, making due allowance for the change in the value of g consequent on a change of latitude. It is found that this pressure is variable at the same place, with or without changes of temperature, and that in ascending mountains, or in any way rising above the level of the place, the pressure diminishes. This is in accordance with the theory of the equilibrium of fluids, for, in ascending, the height of the column of air above the barometer is diminished, and the pressure of the air upon C, which is equal to the weight of the superincumbent column of air, is therefore diminished, and the mercury must descend in the tube.

If then a relation be found between the height of the mercury and the height through which an ascent has been made, it is clear that by observations, at the *same* time, of the barometric columns at two stations, we shall be able to determine the difference of their altitudes.

We shall investigate a formula for this purpose; but it is first necessary to state the laws which regulate the pressures of

the air and gases at different temperatures, and also the laws of the mixture of gases.

71. We have before stated the relation

$$p = k\rho\,(1 + \alpha t)$$

between the pressure, density, and temperature of an elastic fluid: it is deduced from the two following results of experiment:

(1) *If the temperature be constant the pressure of air varies inversely as its volume.* (*Marriotte's and Boyle's Law.*)

(2) *If the pressure remain constant, an increase of temperature of* 1° *C. produces in a mass of air an expansion* ·003665 *of its volume.*

Hence, if p be the pressure, and ρ_0 the density of air, at a temperature zero,

$$p = k\rho_0.$$

Suppose now the temperature increased to t, the pressure remaining the same: the conception of this may be assisted by considering the air to be contained in a cylinder in which a moveable piston fits closely, and has applied to it a constant force, so that an increase of the elastic force of the air would have the effect of pushing out the piston, until the equilibrium is restored by the diminution of density, and consequent diminution of pressure: we shall then have from the 2nd law,

$$\rho_0 = \rho\,(1 + \alpha t),$$

taking ρ as the new density and $\alpha = \cdot 003665$;

$$\therefore\ p = k\rho\,(1 + \alpha t).$$

If p', ρ' be the pressure and density of the same fluid at a temperature t',

$$p' = k\rho'\,(1 + \alpha t'),$$

and

$$\frac{p}{p'} = \frac{\rho}{\rho'}\,\frac{1 + \alpha t}{1 + \alpha t'}.$$

The quantity α is the same for gases of all kinds, but k has different values for different gases, which must of course be determined experimentally in every case.

The pressure of a mixture of different elastic fluids.

72. Consider two different gases, contained in vessels of which the volumes are V and V', and let their pressures and temperatures, p and t, be the same.

Let a communication be established between the two vessels, or transfer both the gases to a closed vessel, the volume of which is $V+V'$: it is found that, unless a chemical action take place, the two gases do not remain separate, but permeate each other until they are completely mixed, and that, when equilibrium is attained, the pressure and temperature are the same as before. From this important experimental fact we can deduce the following proposition.

If two gases having the same temperature be mixed together in a vessel, the volume of which is V, *and if the pressures of the two gases, alone filling the volume* V, *be* p *and* p′, *the pressure of the mixture will be* p + p′.

Suppose the two gases separated; let the gas, of which the pressure is p, have its volume changed, without any alteration of temperature, until its pressure becomes p'; its volume will be, by Marriotte's law, $\frac{p}{p'}V$.

Let the two gases be now mixed in a vessel, of which the solid content is

$$V+\frac{p}{p'}V, \text{ or } \frac{p+p'}{p'}V;$$

the pressure of the mixture will be still p', and the temperature will be unaltered. If the mixture be then compressed into a volume V, its pressure will become, by the application again of Marriotte's law, $p+p'$.

This result is obviously true for a mixture of any number of gases.

73. *Two volumes* V, V′, *of different gases at pressures* p, p′, *respectively, are mixed together, so that the volume of the mixture is* U, *to find the pressure of the mixture.*

The pressures of the two gases, reduced to the volume U, are respectively,

$$\frac{V}{U}p, \quad \frac{V'}{U}p',$$

and therefore, by the preceding article, the pressure of the mixture is

$$\frac{V}{U}p + \frac{V'}{U}p';$$

and if ϖ be this pressure, we have

$$\varpi U = pV + p'V'.$$

74. The laws and results of the preceding articles are equally true of *vapours*, the only difference between the *mechanical* qualities of vapours and gases, irrespective of their chemical characteristics, being that the former are easily condensed into liquids by lowering the temperature, while the latter can only be condensed by the application either of great pressure or extreme cold, or of a combination of both*.

75. If water be introduced into a space containing dry air, vapour is immediately formed, and it is found that the pressure and density of the vapour are dependent only on the temperature, and are quite independent of the density of the air, and indeed are exactly the same if the air be removed. If the temperature be increased or the space enlarged, an additional quantity of vapour will be formed, but if the temperature be lowered or the space diminished, some portion of the vapour will be condensed.

While a sufficient quantity of water remains, as a source from which vapour is supplied, the *space* will be always *saturated* with vapour, that is, there will be as much vapour as the temperature admits of; but if the temperature be so raised that all the water is turned into vapour, then for that, and all higher temperatures,

* Professor Faraday has succeeded in condensing a number of different gases; for instance, carbonic acid, hydrochloric acid, and others requiring a considerable pressure for the purpose, have been reduced to a liquid form. Oxygen, hydrogen, and nitrogen have not yet yielded, but there seems no reason to suppose, if a sufficient pressure can be applied, and a sufficient degree of cold obtained, that these gases will not follow the same law as those which have been liquefied. Such experimental results point to the general conclusion that all gases are merely the vapours of liquids of different kinds.

the pressure of the vapour will follow the same law as the pressure of the air.

In any case, whether the space be *saturated* or not, if p be the pressure of the air, and ϖ of the vapour, the pressure of the mixture is $p + \varpi$.

76. The atmosphere always contains aqueous vapour, the quantity being greater or less at different times; if any portion of the space occupied by the atmosphere be saturated with vapour, that is, if the density of the vapour be as great as it can be for the temperature, then any reduction of temperature will produce condensation of some portion of the vapour, but if the density of the vapour be not at its maximum for that temperature, no condensation will take place until the temperature is lowered below the point corresponding to the saturation of the space.

Formation of Dew. If any surface, in contact with the atmosphere, be cooled down below the temperature corresponding to the saturation of the space near it, condensation of the aqueous vapour will ensue, and the condensed vapour will be deposited in the form of *dew* upon the surface. The formation of dew on the ground depends therefore on the cooling of its surface, and this is in general greater and more quickly effected, when the sky is free from clouds, and when, consequently, the loss of heat by radiation is greater than under other circumstances.

77. Illustrations of the foregoing laws may be obtained by observing the moisture formed on the surface of a glass containing iced liquid, the long trail of steam left behind by a locomotive in rainy weather, the clouds which are formed round mountain peaks, and other natural phænomena of a similar kind.

Whole mass of the Earth's Atmosphere.

78. Some idea may be formed of the mass of air and vapour surrounding the earth by means of the barometer. Supposing the earth to be a sphere of radius r, and that the height of the barometric column, h, is the same at all points of its surface, the

mass of the atmosphere is approximately equivalent to the mass $4\pi\sigma r^2 h$ of mercury.

Let ρ be the mean density of the earth;

$\therefore$ the mass of the atmosphere : the mass of the earth

$$= 4\pi\sigma r^2 h : \rho \frac{4}{3}\pi r^3$$

$$= 3\sigma h : \rho r.$$

But, taking water as the standard substance, $\sigma = 13{\cdot}57$, and ρ has been found to be about $5{\cdot}5$; and, if we take $29{\cdot}9$ inches as an approximate value of h, it will be found that the ratio of the masses is somewhat less than the ratio of one to a million*.

The height of the homogeneous atmosphere.

79. If the whole column of air had the same density throughout as at the surface, its height being l, and h the height of the mercury, we should have

$$\sigma h = \rho l,$$

where ρ is the density of the air. It has been found that the ratio $\sigma : \rho$ is about $10462 : 1$, and therefore, employing as before $29{\cdot}9$ as a value of h, it will be found that l is a little less than 5 miles.

Necessary limit to the height of the atmosphere.

80. It is clear that, since at a distance from the earth's surface its attraction diminishes, and the density and pressure of the air are therefore diminished, the above result is very far from the truth. A *limit* to the height can however be found from the consideration that, beyond a certain distance from the earth's centre, its attraction will be unable to retain the particles of air in the circular

* The observations on the motion of pendulums, made by the Astronomer Royal at the Harton Colliery in 1854, have thrown doubt on the accuracy of the value $5{\cdot}5$, which has been assumed, in Art. 78, as a measure of the mean density of the earth.

The value deduced from the Harton *Observations* is $6{\cdot}566$ with a probable error $\pm .0182$. *Phil. Trans.* 1856.

paths, which they must describe about the earth, in order to remain in a state of relative equilibrium.

At the equator the expression $\omega^2 r$, ω being the earth's angular velocity, is equal to $\frac{g}{289}$, and therefore, at a height z, the force necessary to retain a particle m of air in its circular motion is equal to $\frac{mg}{289}\frac{r+z}{r}$; the earth's attraction at the same height

$$=\frac{mgr^2}{(r+z)^2};$$

and the extreme height is given by the equation

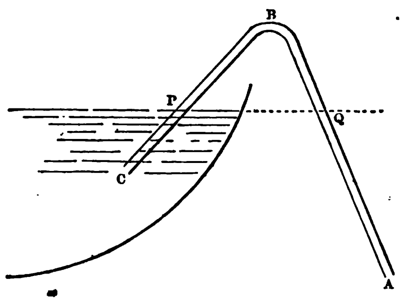
$$\frac{r^2}{(r+z)^2}=\frac{r+z}{289\,r}$$

or
$$z=r\{\sqrt[3]{(289)}-1\},$$

that is, z is a little greater than $5r$.

It is possible however that this height is considerably beyond the true height, for the temperature of the air has been found, by experiments made in balloons, to diminish with great rapidity during an ascent, and it is therefore quite possible, that, at a height less than $5r$, the air may be liquefied by extreme cold, and its external surface would be, in that case, of the same kind as the surfaces of known inelastic fluids.

81. The action of a *siphon* is an illustration of the effect of atmospheric pressure. The siphon consists of a bent tube ABC



open at both ends; if it be filled with water, and the ends closed,

and if the end C be then immersed in water, and the siphon held so that the end A is below the surface of the water, it is clear that the pressure at A will be greater than the pressure at C, and therefore greater than the pressure at P, which is equal to the atmospheric pressure on the surface of the water. If the end A be now unclosed, the water will flow out, and, in consequence of the diminution of pressure in the tube, the atmospheric pressure on the surface will force the water up the tube, and a continuous flow of water results, until the surface is lowered so that the height of B above the surface is greater than the height h of the water-barometer.

If the height of B be originally greater than h, the effect of unclosing the end C will be that the fluid in BP will sink till its surface is at a height h above the surface of the water in the vessel, while the fluid in PA will remain at rest as long as the end A is closed.

In a similar manner the action of a pump depends upon the atmospheric pressure, and is equally limited; and illustrations of Marriotte's law may be found in the contraction of the air-space in a *diving-bell*, consequent on its descent, and in the necessity for pumping into it air from above, in order to prevent the rise of the water within it.

The determination of heights by the barometer.

82. Consider a vertical column of the atmosphere at rest under the action of gravity: at a height z let p be the pressure and ρ the density, and at a height $z + \delta z$, let $p + \delta p$ be the pressure.

If A be the area of the section of the column, the volume $A\delta z$ of air may be considered as in equilibrium under the action of the pressures pA and $(p + \delta p)A$, and of its weight $g\rho A\delta z$.

Hence we have $\qquad \delta p = -g\rho\delta z;$

and, if t be the temperature, $p = k\rho(1 + \alpha t)$;

$$\therefore \text{ in the limit } \frac{k}{p} \cdot \frac{dp}{dz} = -\frac{g}{1+\alpha t},$$

we shall suppose t constant, and therefore

$$k \log p = -\frac{gz}{1+\alpha t} + C,$$

and, if p' be the pressure at a height z', we obtain

$$k \log \frac{p}{p'} = \frac{g(z'-z)}{1+\alpha t}.$$

Let h, h', be the observed heights of the barometer at two stations, the heights of which are z and z'; then, taking σ as the density of mercury at a temperature zero, and τ, τ', as the temperatures at the two stations,

$$p = g\sigma h(1-\theta\tau), \text{ and } p' = g\sigma h'(1-\theta\tau');$$

$$\therefore\ z'-z = \frac{k}{g}(1+\alpha t)\log\frac{h(1-\theta\tau)}{h'(1-\theta\tau')};$$

t may be taken as approximately equal to $\frac{1}{2}(\tau+\tau')$, and we thus have an equation from which the difference of the heights of the two stations can be calculated.

83. If however the heights above the earth's surface be considerable, it is necessary to take account of the variation of gravity at different distances from the earth's centre. We proceed then to an investigation of a more exact formula.

Let g be the measure of gravity at the level of the sea, and r the radius of the earth, then, at a height z, the attractive force is measured by

$$g\frac{r^2}{(r+z)^2},$$

and the equation of equilibrium is

$$dp = -g\frac{r^2}{(r+z)^2}\rho dz;$$

we have also $p = k\rho(1+\alpha t)$, and it is here important to observe that p is the sum of the pressures due to the air itself, and to the aqueous vapour which is mixed with it, so that, if ρ' be the density of the aqueous vapour, p is the sum of two quantities in the form

$$k\rho(1+\alpha t) + k'\rho'(1+\alpha t),$$

and therefore the quantity $k\rho$ in the above equation is the sum of the two $k\rho$, $k'\rho'$, corresponding respectively to the air and the aqueous vapour.

From the two equations above we obtain

$$k\frac{dp}{p} = -\frac{1}{1+\alpha t}\frac{gr^2 dz}{(r+z)^2},$$

and, as before, we shall consider t constant, and equal to the mean of the temperatures at the two stations.

By integration

$$k \log p = \frac{1}{1+\alpha t}\frac{gr^2}{r+z} + C,$$

$$\text{and } \therefore\ k \log\frac{p'}{p} = \frac{gr^2(z-z')}{(1+\alpha t)(r+z)(r+z')} \quad\text{.......... (1).}$$

Let h, h', be the observed heights of the mercury, and τ, τ', the temperatures, as before; then, since the force of gravity at a height z is measured by the quantity $\frac{gr^2}{(r+z)^2}$, we have

$$p = \frac{gr^2}{(r+z)^2}\sigma h(1-\theta\tau),$$

$$p' = \frac{gr^2}{(r+z')^2}\sigma h'(1-\theta\tau'),$$

$$\frac{p'}{p} = \left(\frac{r+z}{r+z'}\right)^2 \frac{1-\theta\tau'}{1-\theta\tau}\frac{h'}{h} \quad\text{.................... (2),}$$

and therefore, observing that θ is a very small quantity,

$$z - z' = \frac{k(1+\alpha t)(r+z)(r+z')}{\mu g r^2}\left\{\log_{10}\frac{h'}{h} + 2\log_{10}\frac{r+z}{r+z'} - \mu\theta(\tau'-\tau)\right\},$$

where $\mu = \log_{10}\epsilon = \cdot 4342945$.

From this formula, if z' be known, the value of z can be calculated.

If the lower station be nearly at the level of the sea, $z'=0$, and

$$z = \frac{k(1+\alpha t)}{\mu g}\left(1+\frac{z}{r}\right)\left\{\log_{10}\frac{h'}{h} + 2\log_{10}\left(1+\frac{z}{r}\right) - \mu\theta(\tau'-\tau)\right\} \ldots (3).$$

84. In the preceding investigation we have taken no account of the variation of gravity at different parts of the earth's surface; but if g' be the measure of gravity at a place of which the latitude

is λ', and g at a place of latitude λ, it has been found, (Poisson, Art. 628), that

$$\frac{g}{g'} = \frac{1 - \cdot 002588 \cos 2\lambda}{1 - \cdot 002588 \cos 2\lambda'}:$$

the value of g obtained from this equation, in which g' and λ' are supposed to be known, must be employed in the above formula.

If λ' be the latitude of Paris, the value of the quantity

$$\frac{k}{\mu g'}(1 - \cdot 002588 \cos 2\lambda') \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

is nearly 18336 French metres or about 60158·56 English feet*, and, representing this numerical quantity by c, the expression for z becomes

$$\frac{c\,(1+\alpha t)\left(1+\frac{z}{r}\right)}{1 - \cdot 00258 \cos 2\lambda}\left\{\log_{10}\frac{h'}{h} + 2\log_{10}\left(1+\frac{z}{r}\right) - \mu\theta\,(\tau' - \tau)\right\} \ldots (5).$$

The value of c may be obtained by direct calculation of the expression (4), and the calculated value is 18337·46 metres; it has been found however, by comparing the results of trigonometrical measurements with the results of the formula (5), that 18336 metres is a more accurate value of the coefficient.

In order to calculate z from the formula (5), an approximate value must be first obtained by neglecting $\frac{z}{r}$ in the right-hand member of the equation; if this approximate value be then employed in the same expression, a more accurate value will result, and the same process may, if necessary, be repeated.

85. Other corrections are however necessary in order to render the determination of heights by the barometer very exact in practice; the value of k for instance is modified by the fact that the density of aqueous vapour at a given temperature and pressure is less than the density of dry air under the same

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* A French metre is 39·37079 inches.

circumstances, and the proportion of aqueous vapour to dry air may be, and in general will be, different at the two stations.

Moreover, if the upper station be on the surface of the ground, the attraction of the portion of the earth which is above its mean level must be taken account of. The effect of this attraction is to increase the quantity $\frac{gr^2}{(r+z)^2}$ by $\frac{3gz}{4r}$, so that, at a height z, the force of gravity is measured by

$$\frac{gr^2}{(r+z)^2}+\frac{3gz}{4r},$$

or, approximately, $g\left\{1-\frac{5z}{4r}\right\}$, (Poisson, *Mécanique*, Art. 629); the equation for p will be in this case

$$dp=-g\left\{1-\frac{5z}{4r}\right\}\rho\, dz,$$

and therefore, if the lower station be at the level of the sea,

$$k(1+\alpha t)\log\frac{p'}{p}=gz\left(1-\frac{5}{8}\frac{z}{r}\right)$$

or

$$z=\frac{k(1+\alpha t)}{g}\left(1+\frac{5}{8}\frac{z}{r}\right)\log\frac{p'}{p}.$$

In place of the equation (2) we shall have

$$\frac{p'}{p}=\left(1+\frac{5}{4}\frac{z}{r}\right)\frac{1-\theta\tau'}{1-\theta\tau}\frac{h'}{h},$$

and the final equation for z will be obtained by substituting in (5),

$$1+\frac{5}{8}\frac{z}{r}\text{ for }1+\frac{z}{r},\text{ observing that }\log\left(1+\frac{5}{4}\frac{z}{r}\right)$$

$$\text{is approximately equal to }2\log\left(1+\frac{5}{8}\frac{z}{r}\right).$$

When $\frac{z}{r}$ is very small, it may be neglected in the formula (5). It has however been found in practice that the results are rendered more accurate, for such cases, by employing, as the value of c, 18393 metres. (Duhamel, p. 259.)

86. In the preceding articles we have supposed the temperature of the air to be constant through the whole of the vertical space between the two stations; if however the difference between the heights be very great, a considerable error may be thus introduced, and formulæ have therefore been constructed in which account is taken, on various hypotheses, of the variation of atmospheric temperature. A formula of this kind is given in Lindenau's Barometric Tables, constructed on the supposition that the temperature diminishes in harmonic progression through a series of heights increasing in arithmetic progression.

It must also be noticed that we have assumed the temperature of the mercury in the barometer to be the same as that of the air surrounding it; but in some cases, as for instance when observations are made in a balloon, the barometer may not remain long enough in the same place to acquire the temperature of the air around it. The temperature of the mercury can, however, be observed by a thermometer the bulb of which is placed in the cistern of the barometer, and the temperatures so obtained must be employed in the equation (2) of Art. (83).

87. The two following problems are illustrative of the principles of this chapter.

(1) A piston without weight fits into a vertical cylinder, closed at its base and filled with atmospheric air, and is initially at the top of the cylinder; water being poured slowly on the top of the piston, find how much can be poured in before it will run over.

Let a be the height of the cylinder, and z the depth to which the piston will sink; then in the position of equilibrium the pressure of the air in the cylinder is $\Pi + g\rho z$, where Π is the atmospheric pressure, and ρ the density of water: but this pressure : Π :: $a : a - z$,

$$\therefore \frac{\Pi a}{a - z} = \Pi + g\rho z.$$

Let h be the height of the water-barometer,

$$\therefore \Pi = g\rho h,$$

$$ha = (a - z)(h + z),$$

and $$z = 0 \text{ or } a - h.$$

Unless then the height of the cylinder is greater than h, no water can be poured in, for, even if the piston be forced down and water then poured on it, the pressure of the air beneath will raise the piston.

The negative solution, when $a < h$, can however be explained as the solution of a different problem leading to the same algebraic equation. Suppose the cylinder to be continued above the piston, and let it be required to raise the piston through a space z by a force which shall be equal to the weight of the cylindrical space z of water.

This leads to the equation

$$\frac{\Pi - g\rho z}{\Pi} = \frac{a}{a+z},$$

$$\text{or} \quad z = h - a.$$

(2) Required to determine the motion of a balloon on the supposition that the mass of air displaced by it in any position is homogeneous, and that the temperature throughout is constant.

Let z be the height of the centre of gravity of the balloon, m its mass, V its volume, and ρ the density of the air at the height z; then the equation which determines the motion is

$$m\frac{d^2z}{dt^2} = g'\rho V - mg',$$

where

$$g' = g\frac{r^2}{(r+z)^2}.$$

But from the equations $dp = -g'\rho dz$ and $p = k\rho$, we obtain

$$p = \Pi\epsilon^{-\frac{grz}{k(r+z)}},$$

and therefore

$$m\frac{d^2z}{dt^2} = \frac{\Pi Vgr^2}{k(r+z)^2}\epsilon^{-\frac{grz}{k(r+z)}} - mg\frac{r^2}{(r+z)^2};$$

from which, putting $m = \sigma V$, multiplying by $2\frac{dz}{dt}$, and integrating,

$$\sigma\left(\frac{dz}{dt}\right)^2 = C - 2\Pi\epsilon^{\frac{-grz}{k(r+z)}} + \frac{2\sigma gr^2}{r+z};$$

initially $$0 = C - 2\Pi + 2\sigma gr,$$

$$\therefore \sigma\left(\frac{dz}{dt}\right)^2 = 2\Pi\left\{1 - \epsilon^{\frac{-grz}{k(r+z)}}\right\} - \frac{2\sigma grz}{r+z}.$$

The greatest height of the balloon is given by putting

$$\frac{dz}{dt} = 0,$$

and, if the mean density of the balloon differ very little from that of the air, $\frac{z}{r}$ will be small, and an approximate value may be found.

EXAMPLES.

1. Two vessels contain air having the same pressure Π but different temperatures t, t'; the temperature of each being increased by the same quantity, find which has its pressure most increased.

If the vessels be of the same size, and the air in one be forced into the other, find the pressure of the mixture at a temperature zero.

2. The temperature of the air in an extensible spherical envelope is gradually raised t^0, and the envelope is allowed to expand till its radius is n times its original length; compare the pressure of the air in the two cases.

3. A cylindrical vessel, closed at both ends, and placed so that its axis is vertical, is half filled with mercury at a temperature $0^\circ C$, the remaining space being occupied by air at the same temperature.

The expansion of mercury between the temperatures 0° and $100^\circ C$ being ·018 of its original volume, and that of air ·3665 of

its original volume for the same pressure, shew that if the temperature be raised to $20^\circ C$ the pressure of the air will be increased in the ratio 1·0772 : 1.

4. If a given body lose in air, when the height of the barometric column is h, the m^{th} part of its weight, find what part of its weight it will lose when the height of the barometric column is h'.

5. The specific gravity of mercury compared with that of water at 68° is 13·568 and at 212° is 13·704. If the expansion of mercury between these points be $\frac{1}{69}$th of its volume at the lower temperature; find that of water between the same points.

6. A faulty barometer indicated 29·2 and 30 inches when the indications of a correct instrument were 29·4 and 30·3 inches respectively; find the length of tube which the air in the tube would fill under the pressure of 30 inches.

7. In an air-pump a leakage takes place during a stroke, by which a quantity of air is admitted, proportional to the difference of the densities of the external and internal air at the beginning of the stroke; find the diminution of density in the receiver in one stroke.

8. If a thermometer, plunged incompletely in a liquid whose temperature is required, indicate a temperature t, and τ be that of the air, the column not immersed being m degrees, prove that the correction to be applied is $\frac{m(t-\tau)}{6840+\tau-m}$, $\frac{1}{6840}$ being the expansion of mercury in glass for 1° of temperature, assuming that the temperature of the mercury in each part is that of the medium which surrounds it.

9. A cylindrical diving bell sinks in water until a certain portion V remains occupied by air, and in this position a quantity of air, whose volume under the atmospheric pressure was $2V$, is forced into it. Shew how far the bell must sink in order that the air may occupy the same space as in the first position.

Find also the condition that when the air is forced in at the first position no air may escape from beneath the bell.

10. A vessel whose sides are vertical is divided into two equal parts by a vertical plane; one part is closed and the other filled with incompressible fluid, and an opening is then made in the partition; compare the heights at which the fluid will stand in the two parts of the vessel, and discuss the several cases which arise from the different positions of the opening.

11. A vessel, in the form of the surface generated by the revolution about its axis of an arc of a parabola terminated by the vertex, is immersed, mouth downwards, in a trough of mercury; shew that the pressure of the air contained in the vessel varies inversely as the square of the distance of the vertex of the vessel from the surface of the mercury within it. Supposing the length of the axis of the vessel to be to the height of the barometer as 45 is to 64, find the depth of the surface of the mercury within the vessel, when the whole vessel is just immersed.

12. Given the height of the mercury in a common barometer, and the depression produced by introducing into it a single drop of water which does not quite all evaporate; find the depression produced by introducing just so much water as will not quite all evaporate into one in which a given length of the upper part of the tube is filled with dry air, and the mercury stands at a given height.

13. The barometer stands at 29·88 inches, and the thermometer is at the Dew Point: a barometer and a cup of water are placed under a receiver, from which the air is removed, and the barometer then stands at ·36 of an inch; find the space which would be occupied by a given volume of the atmosphere, if it were deprived of its vapour without changing its pressure or temperature.

14. A barometer is held suspended in a vessel of water by a string attached to its upper end, so that a portion of the string

is immersed; find the height of the mercury and the tension of the string. If more water be poured into the vessel, how will the tension of the string be affected?

15. A bottle, partly filled with water, is inverted and placed with its mouth just on the surface of a bowl of water, which is being gradually lowered by evaporation: the water from the bottle supplies this loss and keeps the surface at a constant height; find the pressure of the air in the bottle at any time.

16. If the density of the sea varied as the depth below the surface, shew that the tension of the rope supporting a diving bell would be a minimum, if the depth of the surface of the water in the bell below the surface of the sea were $\sqrt{\left(\frac{2\rho h}{\mu}\right)}$; where ρ is the density of mercury, h the height of the barometer, and μ the density of the sea at a depth unity.

17. Assuming the earth to be a sphere revolving about a diameter with an angular velocity ω, shew that its atmosphere, supposed to be a homogeneous gas, can never be in equilibrium with respect to it. Also neglecting all change of temperature, shew that the polar equation to the curve in which a meridian plane cuts a surface of equal pressure is of the form

$$\omega^2 r^3 \cos^2\theta - 2r(ga + k) + 2ga^2 = 0,$$

where a is the radius of the earth, θ the latitude and k a constant.

18. Investigate a formula for determining the difference of the altitudes of two stations, on the supposition that the decrement of temperature in ascending is proportional to the height through which the ascent is made.

CHAPTER VI.

THE TENSION OF FLEXIBLE SURFACES.

88. THE general problem of the equilibrium of flexible surfaces is considered by Lagrange, *Mécanique Analytique*, Tom. I., and also, more fully, by Poisson, *Mémoires de l'Institut*, 1812; it is proposed in this Chapter to discuss one class of the questions which arise out of the general case, those namely which have reference to the action of fluids upon flexible surfaces.

The pressure of a fluid at rest being normal to any surface with which it is in contact, we have, in fact, to consider the equilibrium of flexible surfaces at rest under the action of normal pressures, and of the tensions at their bounding lines.

For the sake of generality the term 'flexible surface' is employed as the representative of substances, such as cloth and paper, which do not offer any sensible resistance to bending, and which, when bent or twisted, do not tend to return to their original form. Perfectly flexible surfaces, whether extensible or inextensible, are therefore to be looked upon as inelastic.

In the following articles we shall suppose that the action between any two portions of a flexible surface is *wholly tangential.*

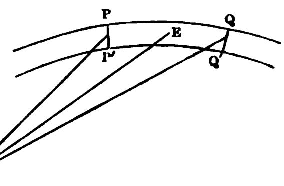
Measure of Tension.

89. Conceive a flexible and inelastic surface, extensible or inextensible, in a state of tension, and let QQ' be a small arc of the section through P made by a normal plane; then if $t \,.\, QQ'$ be the resultant action, perpendicular to QQ' in the tangential plane, between the portions of surface bounded by the line QQ', t is the measure of the tension at P; in other words, t is the rate of tension at P, or the force which would be exerted on a section of the substance, the length of which is unity, in the same state of tension throughout as the surface at P.

It will be seen that we have assumed the resultant tension perpendicular to QQ'; this is not necessarily, or even generally, the case, but it will be found that the definition we have given of the measure of tension is sufficient for the discussion of the problems which present themselves.

90. *A vessel in the form of a circular cylinder, the curved surface of which is flexible, contains fluid; the axis of the cylinder being vertical, it is required to find the relation between the pressure and tension at any point.*

Let PQ' be a small portion of the surface contained between two planes perpendicular to the axis and two generating lines of the cylinder.



Since the pressure is the same at all points in the same horizontal plane, it is clear, from considerations of symmetry, that the tension at all points of the surface in the same horizontal plane will be the same; let t be the horizontal tension and p the pressure, at any point of PQ', and suppose the element PQ' of the surface to be made rigid; then its equilibrium will be maintained by the normal pressure of the fluid, $pPP'.PQ$, the tangential forces tPP' and tQQ', and by the vertical tensions on PQ and $P'Q'$, if there be any tension in the vertical direction.

Hence, resolving the forces in the direction of the normal OE, and employing r for the radius of the cylinder,

$$pPP'PQ = 2tPP' \sin\left(\frac{1}{2}POQ\right),$$

$$= 2tPP' \frac{1}{2} \frac{PQ}{r} \text{ ultimately},$$

or $t = pr$.

91. *If fluid at rest under the action of given forces be contained in a cylindrical surface of any form, the tension at any*

point of a section perpendicular to the axis of the cylinder is the same.

Let PQ', figure, Art. (90), be an element of the surface, O the centre of curvature at E, t the tension at P, $t + \delta t$ at Q, and $\delta\phi$ the angle between the tangents at P and Q.

The fluid pressure on PQ' may be supposed to act along OE*, and therefore, resolving the forces parallel to the tangent at P,

$$(t + \delta t)\cos\delta\phi - t = pPQ\sin\frac{\delta\phi}{2},$$

$$= pr\delta\phi\sin\frac{\delta\phi}{2},$$

if r be the radius of curvature OE:
from this equation we obtain

$$\frac{\delta t}{\delta\phi} = \frac{1}{2}(pr + t)\,\delta\phi + \text{terms involving } (\delta\phi)^2 \ldots\ldots$$

and therefore in the limit,

$$\frac{dt}{d\phi} = 0;$$

and, since the difference of the tensions at any two points of the same perpendicular section $= \int \frac{dt}{\delta\phi}\,d\phi$, between the limiting values of ϕ which define the points, it follows that this difference vanishes, and the tension in the direction perpendicular to the generating lines is therefore constant.

By resolving the forces in the direction OE, we shall obtain, as in the previous article, the relation

$$t = pr,$$

between the tension perpendicular to the generating line, the pressure, and the curvature, at any point of the surface.

92. Ex. A rectangular piece of a flexible and inextensible substance has its sides AB, CD fastened to the sides of a box, and its other sides fit the box closely, so that liquid is contained in it without escaping; required to determine the form of the curve BC.

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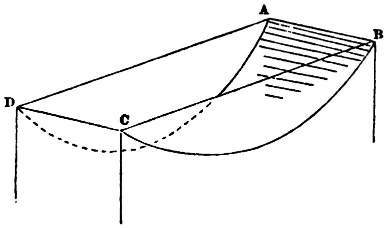
* More exactly, the direction of the fluid pressure lies between OP and OQ, and the resolved part of this force between zero and $pPQ\sin\delta\phi$: the result in either case is the same.

The surface formed is evidently cylindrical, and the tension at any point perpendicular to the direction of generating lines, and constant throughout.

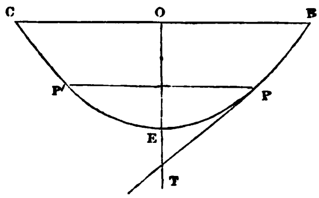


We have then $t = c$, and therefore $c = pr$, if r be the radius of curvature; but if x be the depth below the plane $ABCD$,

$$p = g\rho x,$$

and therefore $$c = g\rho xr.$$

Take the middle point of BC as origin,



then, for the arc BE, $r = \dfrac{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{-\dfrac{d^2y}{dx^2}}$,

since $\dfrac{dy}{dx}$ decreases algebraically as x increases;

$$\therefore\ -\frac{\dfrac{d^2y}{dx^2}}{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}} = \frac{g\rho}{c}x;$$

integrating $$\frac{-\dfrac{dy}{dx}}{\sqrt{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}}} = C + \frac{g\rho}{2c}x^2.$$

If $OE = a$, $\dfrac{dy}{dx}$ is infinite when $x = a$, and we obtain

$$\frac{-\dfrac{dy}{dx}}{\sqrt{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}}}=1+\frac{g\rho}{2c}(x^2-a^2)=\frac{g\rho}{2c}(b^2+x^2) \text{ suppose,}$$

or,
$$\frac{dy}{dx}=\mp\frac{g\rho\,(b^2+x^2)}{\sqrt{\{4c^2-g^2\rho^2(b^2+x^2)^2\}}},$$

which is the differential equation to the curve, the sign being $-$ or $+$ according as x, y, are co-ordinates of P or P', and the constants being determined by the conditions that BC and the arc BEC are of given lengths.

Since $r=\dfrac{t}{p}$, it is clear that the curvature at each of the points B and C is zero.

Let $PTO=\phi$, then $\sin\phi=\dfrac{g\rho}{2c}(b^2+x^2)$, and, if $AB=l$,

$$2lt\cos\phi = \text{the weight of the fluid above } PEP'$$

$$=2g\rho lxy+\int_x^a 2g\rho yl\,dx,$$

$$\therefore \int_x^a 2y\,dx=\sqrt{\left\{\frac{4c^2}{g^2\rho^2}-(b^2+x^2)^2\right\}}-2xy,$$

an expression for the area PEP'.

Hence making $x=0$, and $y=OB$, the area CEB

$$=\sqrt{\left(\frac{4c^2}{g^2\rho^2}-b^4\right)},$$

and the whole volume of fluid is

$$\frac{l}{g\rho}\sqrt{(4c^2-g^2\rho^2b^4)}.$$

If the curve be vertical at B and C, $\dfrac{dy}{dx}=0$ when $x=0$, and therefore $b=0$, or

$$\frac{2c}{g\rho}-a^2=0;$$

$$\therefore\ a=\sqrt{\left(\frac{2c}{g\rho}\right)},$$

and the equation to the curve becomes

$$\frac{dy}{dx}=\frac{x^2}{\sqrt{(a^4-x^4)}}.$$

*

Let $2e$ be the length of the arc BEC, then

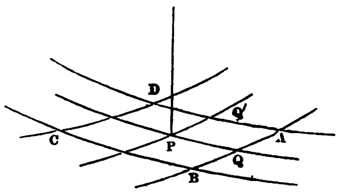
$$e=\int_0^a \sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}\,dx=\int_0^a \frac{a^2dx}{\sqrt{(a^4-x^4)}},$$

an equation by which a, and therefore also c, is defined. The curve thus obtained is called the *Lintearia;* the investigation of its equation was first effected by James Bernoulli*.

93. *A flexible surface of any form is exposed to the action of fluid; required to find the relation between the pressure, tension, and curvature, at any point.*

Let Q, Q', be points contiguous to P, on the lines of curvature PQ, PQ', through P; draw normal planes through Q and Q', perpendicular to the lines, PQ, PQ', cutting the surface in the arcs AB, AD, and let BC, CD, be the arcs of section made by normal planes through contiguous points in QP, $Q'P$, produced. An element of the surface $ABCD$ is thus formed, and the tensions of the sides AB, BC, CD, DA, will be supposed to act at the points on the lines of curvature through which the normals are drawn, and in directions of those lines. Let t, t' be the tensions at P in the directions PQ, PQ', respectively.

The element BD is kept at rest by the tangential force tAB, tCD, $t'AD$, $t'BC$, and the normal force $p.AB.BC$.



Let r, r', be the radii of curvature at P of the curves PQ, PQ'; then, resolving along the normal at P, we have ultimately

$$p.AB.BC=2tAB\frac{\frac{1}{2}AD}{r}+2t'BC\frac{\frac{1}{2}AB}{r'},$$

assuming that the tensions of AB, AD, are respectively perpendicular to those lines;

or $$p=\frac{t}{r}+\frac{t'}{r'}.$$

* The history of this problem is given in Walton's *Hydrostatical Problems*, p. 207.

We have taken the tensions of AB and CD to be ultimately the same; this is evidently the case, for if tAB and $(t+\delta t)\ CD$ be these tensions, their ratio $= t : t + \delta t$, which, if there be no discontinuity of curvature, is ultimately a ratio of equality*.

If the nature of the surface be such that $t' = t$, the above equation is

$$\frac{p}{t} = \frac{1}{r} + \frac{1}{r'},$$

or, if $z = f(x, y)$ be the equation to the surface,

$$\frac{p}{t} \cdot \left\{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right\}^{\frac{3}{2}}$$

$$= \left\{1 + \left(\frac{dz}{dy}\right)^2\right\} \frac{d^2z}{dx^2} - 2\frac{dz}{dx}\frac{dz}{dy}\frac{d^2z}{dx\,dy} + \left\{1 + \left(\frac{dz}{dx}\right)^2\right\} \frac{d^2z}{dy^2};$$

the equation obtained by Lagrange and Poisson.

94. Lagrange, in the discussion of this problem, assumes that the tension at any point of a line on the surface, between the two portions separated by the line, is perpendicular to it, and also that the tension at any point is the same in every direction.

Poisson obtains general equations of equilibrium, and thence reduces, as a particular case, the equation of Lagrange.

The use of the several equations is illustrated, in the *Mémoire*, by an investigation of the catenary, and also by the determination of the equation which defines the small vibrations of a rectangular surface unequally stretched in the directions of its length and breadth.

If it be assumed that the tension at any point of a surface is always perpendicular to a line of division through that point, it can be shewn that the tension at any point is the same in every direction.

Let a small triangular portion of the surface be supposed rigid; then the equilibrium in the tangent plane is entirely determined by the tensions of the sides of the triangle, for the tangential impressed forces, if there be any, will ultimately

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* It does not follow that the tension along a line of curvature is constant; $t + \delta t : t$ is ultimately a ratio of equality; but $t : t + \int dt$ is not so necessarily.

vanish in comparison with the tensions; and since these tensions are perpendicular to the sides, they must be in the ratio of their lengths, and therefore the measures of tension in all directions are the same.

95. *If a flexible surface, of such a nature that the tension of any section is perpendicular to the section, be at rest under the action of fluid pressure, the tension at all points will be the same.*

For any two points P, Q, on a surface may be conceived to be connected by two lines of curvature PE, QE, meeting in E, and it may be shewn, by taking small rectangular elements, as in Art. (91), that the tensions at P and Q are respectively the same as at E, and consequently the tensions at P and Q are equal.

In order to render this reasoning more complete, let t, $t + \delta t$, be the tensions at two points near each other on a line of curvature, and let $\delta\phi$ be the angle between the normals at the two points, then, as in Art. (91), δt is equivalent to an expression of the form $A\,(\delta\phi)^2$, and therefore $\Sigma\,(\delta t)$, which expresses the difference between the tensions at any two points, is of the form

$$\Sigma A\,(\delta\phi)^2, \quad \text{or} \quad \delta\phi \int A\,d\phi, \quad \text{or} \quad B\delta\phi,$$

and therefore vanishes in the limit. The difference between the limiting values of ϕ in the integral is the *whole* angle through which the normal line turns in passing from one point to the other, irrespective of the directions of the planes in which the successive pairs of consecutive normal lines may lie between those two points.

In the case of such surfaces as these, in which t is constant, the equation

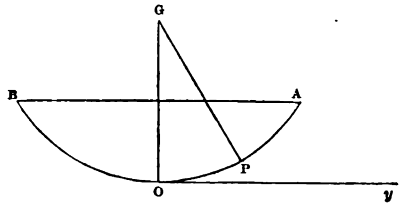
$$\frac{1}{r} + \frac{1}{r'} = \frac{p}{t},$$

where p is a known function of the position of the point on the surface, determines the form of the surface.

Ex. An extensible surface, in form a circular lamina, has its circumference fixed, and is acted upon by the pressure of an elastic fluid; required to find the form it assumes.

In this case, taking the fluid pressure to be constant,

$$\frac{1}{\rho} + \frac{1}{\rho'} \text{ is constant} = \frac{2}{c} \text{ suppose,}$$



and the surface may evidently be assumed to be one of revolution.

Take a tangent line parallel to the plane of the lamina as the axis of y;

$$\text{then, since } \rho = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{-\frac{d^2y}{dx^2}};$$

and $$\rho' = PG = y\sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}.$$

The equation to the curve AOB, changing the variable to y, is

$$\frac{1}{y}\frac{\frac{dx}{dy}}{\sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}}} + \frac{\frac{d^2x}{dy^2}}{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}^{\frac{3}{2}}} = \frac{2}{c},$$

or $$\frac{y\frac{dx}{dy}}{\sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}}} = \frac{y^2}{c} + C:$$

but, at the origin, y and $\frac{dx}{dy}$ vanish, $\therefore\ C = 0$, and the equation is

$$\frac{dx}{ds} = \frac{y}{c},$$

the curve is therefore a circle, as might have been anticipated.

96. As a matter of fact, all flexible surfaces, whether extensible or inextensible, are capable of sustaining tension between

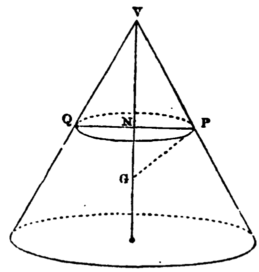
two portions, in a direction not perpendicular to the separating line; and indeed the conception of such a surface as that contemplated in the three preceding articles is of the same nature as the conception of a perfectly rigid body or of a perfect fluid; it must therefore be considered in the discussion of problems on the equilibrium of flexible surfaces, that a disruption will not necessarily take place, when circumstances of the equilibrium require that in certain directions there should be tension in a direction not perpendicular to the line of section.

If sections of a surface be taken along lines of curvature through a point, and if the tensions at points of these lines be perpendicular to them, but unequal, it can be easily shewn that for any other section through the point, the tension is *not* perpendicular to the section.

In illustration of this case the following problem will be considered.

97. A conical perfectly flexible and elastic bag attached, mouth downwards, by the rim to a horizontal plane, and filled with liquid by a small hole at the apex, has, when at rest, the figure of a right circular cone; find the equation to the figure it will assume when detached and the liquid let out, neglecting its weight.

Let t be the tension at P in the direction perpendicular to the generating line VP, t' the tension in the direction VP, and 2α the vertical angle of the cone.



Then $$p = \frac{t}{r} + \frac{t'}{r'} \text{ gives, if } VN = x,$$

$$g\rho x = \frac{t}{PG} = \frac{t}{x \tan\alpha \sec\alpha},$$

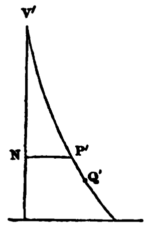
or $$t = g\rho x^2 \tan\alpha \sec\alpha.$$

But $2\pi PNt' \cos\alpha =$ the resultant vertical pressure on VPQ

$$= \frac{2}{3} g\rho\pi x^3 \tan^2\alpha;$$

$$\therefore\ t' = \frac{1}{3} g\rho x^2 \tan\alpha \sec\alpha.$$

Let $V'P'Q'$ be the generating curve of the surface of revolution into which the surface forms itself after the liquid has been let out, $V'N = \xi$, $P'N = \eta$, P' corresponding to the point P.



If $P'Q' = \delta s$, a small arc of the curve,

$$\delta x \sec\alpha = \delta s\left(1 + \frac{t'}{\lambda'}\right),$$

$$\text{and}\quad x\tan\alpha = \eta\left(1 + \frac{t}{\lambda}\right),$$

taking the modulus of elasticity different in the two directions. Taking account of the values of t and t' obtained above, x can be eliminated between these two equations, and the relation between ξ and η will result.

From the first equation, putting $\dfrac{g\rho \tan\alpha \sec\alpha}{3\lambda'} = \dfrac{1}{a^2}$,

$$\frac{ds}{dx}\cos\alpha = \frac{1}{1 + \dfrac{x^2}{a^2}};$$

$$\therefore\ \frac{s}{a}\cos\alpha = \tan^{-1}\frac{x}{a},\ \text{measuring } s \text{ from } V',$$

or $$x = a\tan\left(\frac{s}{a}\cos\alpha\right).$$

Substituting this expression for x in the second equation, we obtain

$$a\tan\alpha\tan\left(\frac{s}{a}\cos\alpha\right) = \eta\left\{1 + \frac{g\rho a^2 \tan\alpha\sec\alpha}{\lambda}\tan^2\left(\frac{s}{a}\cos\alpha\right)\right\},$$

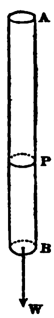
as the differential equation to the curve*.

* If $\lambda = \lambda'$, the equation is

$$a\tan\alpha = \eta\left\{\cot\left(\frac{s}{a}\cos\alpha\right) + 3\tan\left(\frac{s}{a}\cos\alpha\right)\right\}.$$

It will be seen, *a priori*, that t' is not constant, for no rectangular element can be taken on the surface the tensions of the edges of which are perpendicular to the directions of the edges. If, for instance, we consider an element of the surface bounded by two generating lines near each other, and by two contiguous circular sections, the tensions will be perpendicular to the sides, but, the element not being a rectangle, we shall find, in resolving the forces along one generating line, that the tension of the other will enter into the equation, and will not vanish in the limit, in comparison with the tensions of the circular sections.

98. We have hitherto considered only laminæ of uniform thickness, but, in order to include cases in which the lamina is of variable thickness, a more general measure of the tension can be given.



Suppose a bar AB of any homogeneous material to support a weight W, and let κ be the area of the section of the bar; then the tension at the section through P supports W and the weight of the bar PB; and if $\tau\kappa$ is equal to the sum of these weights, τ is the measure of the tension at P.

It will be seen that τ is one dimension lower than the t of Art. (89).

In fact, if e be the thickness of a flexible lamina at any point, the tension at which, measured in the usual way, is t, we have

$$t\delta s = \tau e\delta s,$$

$$\text{or } t = e\tau.$$

Ex. A hollow spherical shell of very small thickness e is formed of the same substance as the bar AB, and W is the greatest weight which AB can support, neglecting its own weight; it is required to find the greatest pressure of elastic fluid which the shell can sustain.

Let r be the radius of the shell;

$$\therefore\ 2t = pr, \quad \text{or } 2e\tau = pr;$$

but $$W = \tau\kappa, \quad \therefore \; p = \frac{2e}{r}\frac{W}{\kappa},$$

which is the the greatest possible pressure.

99. The investigations of this chapter will not in general be applicable to surfaces which are inflexible, or of imperfect flexibility, but, if in any particular case the action between adjacent portions of a surface be wholly tangential, the relations obtained between the tension and the normal pressure will hold good.

For instance, if a vertical circular cylinder formed of any inflexible substance be filled with fluid, the action at any point will be wholly tangential and of the nature of tension.

EXAMPLES.

1. Supposing the cylinders of a Bramah's Press made of the same material, and the thickness of the smaller just sufficient to prevent it from bursting, what must be, at least, the thickness of the larger?

2. A cylindrical vessel is formed of metal a inches thick, and a bar of this metal of which the section is A square inches, will just bear a weight W without breaking. If the cylinder be placed with its axis vertical, find how much fluid can be poured into it without bursting it.

3. A hollow cone, the vertex of which is downwards, is filled with fluid; find where the horizontal tension is greatest.

Also find where the tension in the direction of a generating line is greatest.

4. The top of a rectangular box is closed by an uniform elastic band, fastened at two opposite sides, and fitting closely to the other sides; the air being gradually removed from the box, find the successive forms assumed by the elastic band, and when it just touches the bottom of the box, find the difference between the external and internal atmospheric pressures.

5. An elastic tube of circular bore is placed within a rigid tube of square bore which it exactly fits in its unstretched state,

the tubes being of indefinite length; if there be no air between the tubes and air of any pressure be forced into the elastic tube, shew that this pressure is proportional to the ratio of the part of the elastic tube that is in contact with the rigid tube to the part that is curved.

6. A small uniform flexible tube is inextensible in length, but the perimeter of any transverse section of it follows the ordinary law of extension of elastic strings; if it be filled with fluid and held with its axis vertical, shew that for some distance from the highest point it will appreciably coincide with the surface generated by a rectangular hyperbola revolving about its asymptote.

7. A flexible and elastic envelope without weight, filled with fluid, hangs by one point: find the differential equation to a vertical section of the surface through the point of support.

CHAPTER VII.

THE EQUILIBRIUM OF REVOLVING FLUID, THE PARTICLES OF WHICH ARE MUTUALLY ATTRACTIVE.

100. If a fluid mass, the particles of which attract each other according to a definite law, revolve uniformly about a fixed axis, it is conceivable that, for a certain form of the free surface, the fluid particles may be in a state of relative equilibrium; since however the resultant attraction of the mass upon any particle depends in general upon its form, which is unknown, a complete solution of the problem cannot be obtained.

For any arbitrarily assigned law of attraction, the question is one of purely abstract interest, and it is only when the law is that of gravitation that it becomes of importance, from its relation to one of the problems of physical astronomy.

We shall consider the fluid homogeneous, and confine our attention to two cases; in the first of these the attractive forces are supposed to vary directly as the distance, and, in the second, to follow the Newtonian law.

101. *A homogeneous fluid mass, the particles of which attract each other with a force varying directly as the distance, rotates uniformly about an axis through its centre of gravity; required to determine the form of the free surface.*

The resultant attraction on any particle is in the direction of, and proportional to, the distance of the particle from the centre of gravity; and if μ be a measure of the whole mass of fluid, μx, μy, μz, may represent the components of the attraction, parallel to the axes, on a particle of fluid about the point x, y, z.

Taking the origin at the centre of gravity, and axis of rotation as the axis of z, the equation of equilibrium is

$$dp = \rho\,\{(\omega^2 x - \mu x)\,dx + (\omega^2 y - \mu y)\,dy - \mu z\,dz\};$$

and therefore

$$p = C + \frac{\rho}{2}\{(\omega^2 - \mu)(x^2 + y^2) - \mu z^2\}.$$

At the free surface p is zero or constant, and the equation to the free surface is

$$\left(1 - \frac{\omega^2}{\mu}\right)(x^2 + y^2) + z^2 = D,$$

the constant D depending upon ω, and upon the mass of the fluid.

If $\omega^2 < \mu$, the free surface is a spheroid, which becomes more oblate as ω increases, and when $\omega^2 = \mu$, the free surface consists of two planes; to render this possible we may conceive the fluid enclosed within a cylindrical surface, the axis of which coincides with the axis of rotation.

When $\omega^2 > \mu$, the free surface is a hyperboloid of two sheets, which for a certain value (ω') of ω becomes a cone, the fluid filling the space between the cone and the cylinder. Taking account of the volume of the fluid, the value of ω' can be determined by putting $D = 0$, since the pressure in this case vanishes at the origin.

If $\omega > \omega'$, the surface is a hyperboloid of one sheet, which, as ω increases, approximates to the form of a cylinder, and it is therefore necessary, for large values of ω, to conceive the containing cylinder closed at its ends.

The results of this article, it may be observed, are equally true of heterogeneous fluid, whatever be the law of variation of density in the successive strata.

102. *A mass of homogeneous fluid, the particles of which attract each other according to the Newtonian law, rotates uniformly, in a state of relative equilibrium, about an axis through its centre of gravity; required to determine a possible form of the surface.*

For the reason previously mentioned a direct solution of this problem cannot be obtained, but it can be shewn that an oblate spheroid is a possible form of equilibrium.

Let the equation to the spheroid be

$$\frac{z^2}{c^2}+\frac{x^2+y^2}{c^2(1+\lambda^2)}=1,$$

the axis of rotation being the axis of z.

Then the resultant attractions, towards the origin, on a particle at the point (x, y, z) will be represented by

$$X=\frac{2\pi\rho x}{\lambda^3}\{(1+\lambda^2)\tan^{-1}\lambda-\lambda\},$$

$$Y=\frac{2\pi\rho y}{\lambda^3}\{(1+\lambda^2)\tan^{-1}\lambda-\lambda\},$$

$$Z=\frac{4\pi\rho z}{\lambda^3}\{\lambda-\tan^{-1}\lambda\}(1+\lambda^2),$$

parallel, respectively, to the axes*.

We have then for the surfaces of equal pressure, putting ϵ for

$$\frac{\omega^2}{4\pi\rho},$$

$$\{2\epsilon\lambda^3+\lambda-(1+\lambda^2)\tan^{-1}\lambda\}(xdx+ydy)$$
$$+2(\tan^{-1}\lambda-\lambda)(1+\lambda^2)zdz=0.$$

But, from the equation to the spheroid,

$$xdx+ydy+(1+\lambda^2)zdz=0,$$

and, as these equations must be identical,

$$2\epsilon\lambda^3+\lambda-(1+\lambda^2)\tan^{-1}\lambda=2(\tan^{-1}\lambda-\lambda);$$

an equation the roots of which determine the possible values of λ.

It may be written

$$\frac{3\lambda+2\epsilon\lambda^3}{3+\lambda^2}-\tan^{-1}\lambda=0, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\quad (\alpha),$$

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* These expressions will be found in Laplace's *Mécanique Céleste*, Poisson's *Mécanique*, Duhamel's *Mécanique*, and Todhunter's *Statics*. In the last named, the equation to the spheroid is $\frac{x^2+y^2}{a^2}+\frac{z^2}{a^2(1-e^2)}=1$, but the expressions used in the text will result from the expressions there given by putting $1+e^2=\frac{1}{1+\lambda^2}$.

By the use of λ, irrational quantities are avoided.

and the question is reduced to the discussion of the roots of this equation.

For this purpose consider the curve

$$y = \frac{3x + 2\epsilon x^3}{3 + x^2} - \tan^{-1} x \,; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\beta).$$

The abscissæ of the points where this curve cuts the axis will be the values of λ required.

It must be observed that, in the equation (α), $\tan^{-1}\lambda$ is the least positive angle whose tangent is λ; we have therefore only to consider one branch of the curve (β).

If the signs of x and y be changed, the equation is unaltered; the curve is therefore the same in the compartment $-x$, $-y$, as in $+x$, $+y$, and it is sufficient to examine the nature of the positive portion of the branch.

When $x = 0$, $y = 0$, and as x increases from zero, y begins by being positive, and when x increases indefinitely, has always positive values; hence the curve cuts the axis of x in an even number of points, exclusive of the origin.

Again, $$\frac{dy}{dx} = \frac{2x^2 \{\epsilon x^4 + 2(5\epsilon - 1)x^2 + 9\epsilon\}}{(1 + x^2)(3 + x^2)^2},$$

$\frac{dy}{dx}$ is therefore zero at the origin (a point of inflection), and also at the points given by

$$\epsilon x^4 + 2(5\epsilon - 1)x^2 + 9\epsilon = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\gamma).$$

If the values of x^2, obtained from this equation, be real, and positive, there will be a maximum and a minimum value of y; the former, corresponding to the smallest root, will evidently be positive, since y begins by being positive; if the latter, corresponding to the greatest root, be negative or zero, there will be two zero values of y or one only, and consequently two possible spheroidal forms of equilibrium, or one only.

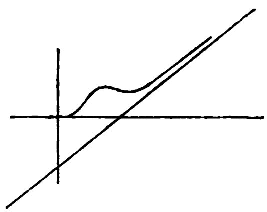
If the minimum value of y be positive, there will be no zero value of y; that is, the equilibrium of the fluid in the form of a spheroid is impossible.

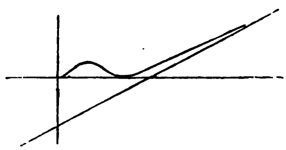
103. The preceding investigation may be illustrated by tracing the curve (β) for different values of ϵ.

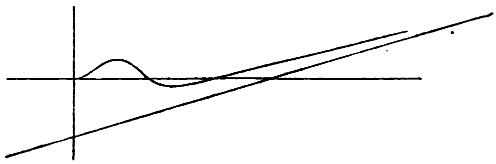
Putting $\tan^{-1} x = \frac{\pi}{2} - \tan^{-1} \frac{1}{x}$, and expanding, we obtain

$$y = 2\epsilon x - \frac{\pi}{2},$$

as the asymptote of the branch of the curve under consideration, and the appended figures exemplify the different cases above mentioned.







Numerical Calculation.

104. *To calculate the limiting value of ω for which the spheroidal form is possible.*

The equation (γ) may have positive roots if $5\epsilon < 1$; moreover the values of x^2 will be real, and positive, if

$$(1 - 5\epsilon)^2 > 9\epsilon^2, \text{ or } 1 - 5\epsilon > 3\epsilon;$$

$$\text{i. e. } \epsilon < \frac{1}{8}.$$

The superior limiting value of ϵ can however be obtained very approximately from the condition that, in the extreme case of possibility, the minimum value of y is zero.

We have then $y = 0$ and $\frac{dy}{dx} = 0$, simultaneously.

Hence, substituting in (β) the value of ϵ obtained from (γ), and putting $y = 0$, we have

$$\frac{x(7x^4 + 30x^2 + 27)}{(x^2 + 1)(x^2 + 9)(x^2 + 3)} - \tan^{-1} x = 0,$$

or
$$\frac{x(7x^2 + 9)}{(x^2 + 1)(x^2 + 9)} - \tan^{-1} x = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ (\delta).$$

An approximate value of the positive root of this equation will be a value of x, which, substituted in (γ), will give approximately the superior limit of the value of ϵ.

Since $\omega^2 = 4\pi\rho\epsilon$ this determines the greatest possible rate of rotation consistent with the existence of a spheroidal form.

When ω is less than the limiting value thus obtained, there will be two spheroids, either of which will be a possible form of the rotating fluid.

105. *Approximate determination of the positive root of the equation*

$$\frac{x(7x^2 + 9)}{(x^2 + 1)(x^2 + 9)} - \tan^{-1} x = 0.$$

Denoting the first member by $f(x)$, it will be found that

$$f'(x) = \frac{8x^4(3 - x^2)}{(x^2 + 1)^2(x^2 + 9)^2}:$$

this is positive from $x = 0$ to $x = \sqrt{3}$, and is afterwards negative;

$f(x)$ therefore increases until $x = \sqrt{3}$, and then diminishes; and, since $f(0) = 0$, $f(x)$ begins by being positive.

By the use of the formulæ

$$\tan^{-1} 2 = \frac{\pi}{4} + \tan^{-1}\frac{1}{3},$$

$$\tan^{-1} 3 = \frac{\pi}{4} + \tan^{-1}\frac{1}{2},$$

it will be found without much difficulty that the root lies between 2 and 3, but the application of Newton's method with the value 2 as an approximate one shews that a closer limit will be convenient.

If then 2.5 be substituted we obtain, by the aid of the formula

$$\tan^{-1}(2.5) = \tan^{-1}(2) + \tan^{-1}\frac{1}{12},$$

$$f(2.5) = \cdot 0025 \text{ approximately.}$$

Let $$x = 2.5 + y,$$

then, approximately, $$y = -\frac{f(2.5)}{f'(2.5)},$$

but $$f'(2.5) = -.085, \text{ nearly};$$

$$\therefore\ y = .0293 \text{ and } x = 2.5293.$$

The substitution of this value in (γ) will give

$$\epsilon = .1123,$$

as the greatest possible of ϵ or $\dfrac{\omega^2}{4\pi\rho}$.

Hence, when ω is such that $\epsilon < .1123$, there are two spheroidal forms of equilibrium.

106. If ϵ is very small, one of the values of x (i. e. λ) will be very small and the other large, and therefore as ϵ decreases, the one spheroid becomes *very oblate* and approximates to a plane lamina, while the other approaches to the form of a sphere, this latter being the form of stable equilibrium.

To find the small value of λ which satisfies the equation

$$\frac{3\lambda + 2\epsilon\lambda^3}{3+\lambda^2} - \tan^{-1}\lambda = 0,$$

expand in ascending powers of λ, and we obtain

$$\lambda^2 = \frac{15\epsilon}{2} \text{ approximately.}$$

This gives a spheroid very slightly oblate, the ratio of its axes being $\sqrt{(1+\lambda^2)} : 1$, or very nearly $1 + \dfrac{15\epsilon}{4} : 1$.

The *large* value of λ is obtained by putting

$$\tan^{-1}\lambda = \frac{\pi}{2} - \tan^{-1}\frac{1}{\lambda},$$

and expanding in powers of $\frac{1}{\lambda}$, a process which gives

$$\lambda = \frac{\pi}{4\epsilon} - \frac{8}{\pi} + \text{terms involving positive powers of } \epsilon,$$

as an approximation.

107. *Application to the case of a fluid, the density of which is equal to the Earth's mean density.*

If r be the Earth's radius and ρ the mean density of the Earth,

$\frac{4}{3}\pi\rho r$ is the attraction at the surface of a sphere of fluid of the same radius as the Earth, and of density ρ.

Suppose for a moment that the Earth is homogeneous, and spherical,

then $\frac{4}{3}\pi\rho r$ measures the force of gravity at the pole.

But, since $\epsilon = \frac{\omega^2}{4\pi\rho}$, and therefore $3\epsilon = \frac{\omega^2 r}{\frac{4}{3}\pi\rho r}$,

$3\epsilon : 1 ::$ difference of the measures of gravity at the pole and the equator : gravity at the pole (g).

Taking a second and a foot as the units of time and space, $g=32$ approximately, $r = 4000 \times 1760 \times 3$, and it will be found that the time of rotation, $\frac{2\pi}{\omega}$, given by the limiting value .1123 of ϵ, is a little more than $\frac{1}{10}$th of a day.

This then is the smallest time in which a homogeneous fluid mass, of density equal to the Earth's mean density, could rotate uniformly so as to be spheroidal in form.

Application to the figure of the Earth.

108. The Earth, as is known by geodetic measurements, differs very slightly in its form from a sphere, and we can therefore apply our equations with great ease to the question of the

homogeneity of the Earth, assuming it to have taken its present form when in a state of fluidity, or to be now a mass of fluid contained within a comparatively thin crust.

It has been found by observation, that for the Earth the ratio $\omega^2 r : g$ is about $1 : 289$, and we have therefore

$$3\epsilon = \frac{1}{289}.$$

But from Art. (106), $\lambda^2 = \frac{15\epsilon}{2} = \frac{5}{579}$,

and the ratio of the axes of the spheroid

$$= 1 + \frac{\lambda^2}{2} : 1 = 232 : 231, \text{ nearly.}$$

This result does not accord with the facts obtained by actual measurement, which give 301 : 300 as an approximate value of the ratio.

The inference is that the Earth is not homogeneous.

109. The foregoing articles are taken chiefly from Laplace, *Mécanique Céleste*, Tome II.

It must be observed that the general problem of the form of a rotating fluid is not solved; all that is shewn being that, in certain cases, an oblate spheroid is a possible form of equilibrium.

If ω be such that $\epsilon > .1123$, it does not follow that equilibrium is impossible, but only that the spheroidal form cannot exist for that particular angular velocity.

Laplace has shewn that a prolate spheroid is not a possible form of equilibrium*: but it has been demonstrated by Jacobi that an ellipsoid with its three axes unequal is a possible form. On this latter point a discussion will be found in the first volume of the *Cambridge Mathematical Journal*, and also in a paper by Mr Ivory, in the *Philosophical Transactions* for 1838.

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* *Méc. Céleste*, Tom. II. p. 69. The proof is also given in Pentécoulant's *Système du Monde*, Tom. II. p. 401.

110. An important distinction has been pointed out by Poisson (Tome II. p. 547), between the surfaces of equal pressure in a fluid at rest under the action of extraneous forces, and in a fluid at rest, or revolving uniformly about a fixed axis, under the action of the mutually attractive forces of its particles.

Let *ABC* be the free surface, and *DEF* any surface of equal pressure; then, in the former case, the resultant force at any point of *DEF* is perpendicular to the surface at that point, and is unaffected by the existence of the fluid between *ABC* and *DEF*; this fluid could therefore be removed without affecting the equilibrium of the fluid mass bounded by *DEF*. In the latter case, the force at any point of *DEF*, although perpendicular to the surface at that point, is the resultant of the attractions of the mass of fluid contained by *DEF*, and of the mass contained between *DEF* and *ABC*; these two components of the resultant force are not necessarily perpendicular to the surface, and the fluid external to *DEF* cannot in general be removed without affecting the equilibrium of the remainder.

111. If, however, the fluid be homogeneous, and the particles attract each other according to the Newtonian law, so that the free surface may be spheroidal, the surfaces of equal pressure will be similar spheroids; and in this case, since the resultant attraction of an ellipsoidal shell on an internal particle is zero, the portion of fluid between *ABC* and *DEF* may be removed, provided the rate of rotation remain unaltered.

Moreover we have shewn, Art. (103), that for a given value of ω not exceeding a determined limit, there are two possible spheroidal forms: let *ABC*, the free surface, have one of these forms, and describe within the fluid mass a concentric spheroid, *GHK*, similar to the other spheroid; then the fluid between *ABC* and *GHK* may be removed without affecting the fluid mass *GHK*.

The action of the shell upon a particle at a point *P* of the surface *GHK* is not perpendicular to the surface at *P*, but this action, combined with the attraction of the mass *GHK*, and the hypothetical force measured by $\omega^2 r$, is perpendicular to the surface, at *P*, of the spheroid passing through *P*, which is concentric with, and similar to, the surface *ABC*.

112. If a fluid mass be set in motion, about an axis through its centre of gravity, with an angular velocity such as to make the value of ϵ greater than the limit obtained in Art. (105), it does not follow that the fluid cannot be in equilibrium in the form of a spheroid, for it may be conceived that the mass will expand laterally with reference to the axis, taking a more flattened shape, until its angular velocity is so far diminished as to render the spheroidal form possible.

If the mass consist of *perfect* fluid, its form will oscillate through the spheroid of equilibrium, but if, as is the case in all known fluids, friction be called into play by the relative displacement of the particles, the oscillations will gradually diminish and at length a position of equilibrium will be attained. By D'Alembert's principle, the 'Angular momentum' of the system, relative to the axis, will remain constant, and this property of the motion enables us to determine the final angular velocity, and the form ultimately assumed*.

Considering the question generally, suppose the mass of fluid set in motion in any way, and then left to itself; the centre of gravity will be either at rest or moving uniformly in a straight line, and all we have to consider is the motion relative to the centre of gravity.

Draw through the centre of gravity the plane, in the direction of which the angular momentum is a maximum; then, however during the subsequent motion the fluid particles act on each other, this plane, which may be called the 'momental' plane, will remain fixed, and when the motion of the particles relative to each other has been destroyed by their mutual friction, the axis perpendicular to this plane will be the axis of rotation of the fluid mass in its state of relative equilibrium.

113. Let H be the given angular momentum of the system, and ω its ultimate angular velocity.

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* The angular momentum of a system, relative to an axis, is the sum of the moments of the moments of the several particles of the system about the axis.

The constancy of the angular momentum is therefore the expression of the principle of 'the conservation of areas.'

Taking c and $c\sqrt{(1+\lambda^2)}$ for the axes of the spheroid of equilibrium, and M for the mass, the expression for the angular momentum is $\frac{1}{2}\cdot\frac{2}{5}Mc^2(1+\lambda^2)\,\omega$;

$$\therefore\ \frac{1}{5}Mc^2(1+\lambda^2)\,\omega = H;$$

we have also
$$\frac{4}{3}\pi\rho c^3(1+\lambda^2) = M,$$

and from these two equations, combined with the equation,

$$\frac{3\lambda+2\epsilon\lambda^3}{3+\lambda^2} - \tan^{-1}\lambda = 0 \ldots \text{Art. (102)},$$

the values of c, ω, and λ can be determined.

From the first two, we obtain

$$\epsilon = \frac{\omega^2}{4\pi\rho} = \frac{25H^2\left(\frac{4}{3}\pi\rho\right)^{\frac{1}{3}}}{3M^{\frac{10}{3}}}(1+\lambda^2)^{-\frac{4}{3}},$$

$$= p\,(1+\lambda^2)^{-\frac{4}{3}}, \text{ suppose};$$

$$\therefore\ \frac{3\lambda+2p\lambda^3(1+\lambda^2)^{-\frac{4}{3}}}{3+\lambda^2} - \tan^{-1}\lambda = 0,$$

is the equation which determines λ.

The left-hand member of this equation is positive when λ is very small, and negative when λ is indefinitely large, and the equation has therefore a positive root; consequently, the fluid mass will at length attain a spheroidal form of equilibrium.

It can be shewn moreover that the equation has only one positive root, and therefore there is one spheroidal form, and one only, towards which the oscillating fluid mass continually approximates.

This discussion is taken from the *Mécanique Céleste*, Tom. II. p. 71, and from *Pentécoulant's Système du Monde*, Tom. II. p. 409.

HYDRODYNAMICS.

CHAPTER VIII.

THE EQUATIONS OF MOTION.

114. We have assumed, as a fundamental property, that "the pressure of a fluid is always normal to any surface with which it may be in contact," and, from this assumption, it follows necessarily that the pressure at any point is the same in all directions. So far as the equilibrium of fluids is concerned, the results of calculation are in accordance with facts, and the above principle appears to be established.

In considering, however, the motion of fluids, very little observation is sufficient to shew that the hypothesis of fluid pressure being normal to surfaces with which the fluid is in contact is no longer tenable as a matter of fact. For instance, if water in a cup be set rotating, the motion is gradually diminished, and in a short time ceases altogether; this can only result from a tangential action between the surface of the cup and the particles of water immediately in contact with it, and from a transmission of this tangential action throughout the whole mass. Other instances might be adduced, such as the action of the air upon pendulums, and the motion of water flowing down sloping tubes, in which the results of calculation and of observation are considerably at variance with each other; but, as the motion of *perfect* fluids only will be discussed in the present treatise, it will be sufficient for our purpose to recognise the fact that the internal friction of fluids produces, in general, a sensible modification of their motions, and the consequent probability that a discrepancy will exist between the conclusions of theory and experiment.

115. The conception of a perfect fluid implies that its pressure on a surface is in all cases a normal pressure, and it follows

necessarily that, whether the fluid be at rest or in motion, *the pressure at any point is the same in all directions.*

Conceive a small tetrahedron of the fluid solidified; then, by D'Alembert's principle, the pressures on the faces, the moving forces arising from external attractions, and forces equal and opposite to the effective moving forces, form a system in equilibrium. Suppose the tetrahedron indefinitely diminished; the extraneous forces, and the effective moving forces, vary as the cubes, while the pressures vary as the squares, of homologous lines, and therefore the former are, in the limit, evanescent compared with the latter. The tetrahedron is, therefore, ultimately at rest under the action of the pressures only, and hence it follows, as in Art. (7), that the pressure is the same in every direction.

116. Prop. *To find the equations of motion.*

Let x, y, z, be the co-ordinates of a point within the fluid in motion, and, at the time t, let u, v, w, be the velocities, parallel to the axes, at the point: u, v, w are therefore, generally, functions of x, y, z, and t.

The velocity *at* any point in a fluid may be looked upon as the velocity of the particle or element of fluid which happens, at the time, to contain the point; or, if the fluid be conceived as made up of ultimate molecules, the velocity at any point is the mean of the velocities of all the molecules contained in an element of fluid about the point, when the element is indefinitely diminished.

Let m be the mass of an element of fluid about the point x, y, z, and let mX, mY, mZ, be the impressed forces acting upon m.

Considered as defining the position of m, x, y, and z are functions of t, and of the quantities defining the initial position of m, and, on this supposition, we have

$$u=\frac{Dx}{Dt}, \quad v=\frac{Dy}{Dt}, \quad w=\frac{Dz}{Dt},$$

employing D as the symbol of differentiation, in order to distinguish between the variations of x, y, z, which depend on the

motion, and the arbitrary variations which are considered in the equilibrium equations we shall have to employ.

The effective forces are

$$m\frac{D^2x}{Dt^2},\quad m\frac{D^2y}{Dt^2},\quad m\frac{D^2z}{Dt^2},$$

or

$$m\frac{Du}{Dt},\quad m\frac{Dn}{Dt},\quad m\frac{Dw}{Dt},$$

and, by D'Alembert's principle, the aggregate of these forces reversed, would, in combination with the impressed forces, maintain the equilibrium of the fluid.

Hence, if p be the pressure, we obtain from Art. (16) the equations

$$\left.\begin{aligned}\frac{dp}{dx}&=\rho\left(X-\frac{Du}{Dt}\right),\\ \frac{dp}{dy}&=\rho\left(Y-\frac{Dv}{Dt}\right),\\ \frac{dp}{dz}&=\rho\left(Z-\frac{Dw}{Dt}\right).\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots(1).$$

The symbol d referring to partial differentiation, we have, since u is a function of x, y, z, and t,

$$\frac{Du}{Dt}=\frac{du}{dt}+\frac{du}{dx}\frac{Dx}{Dt}+\frac{du}{dy}\cdot\frac{Dy}{Dt}+\frac{du}{dz}\cdot\frac{Dz}{Dt},$$

or

$$\frac{Du}{Dt}=\frac{du}{dt}+u\frac{du}{dx}+v\frac{du}{dy}+w\frac{du}{dz},$$

with similar equations for $\dfrac{Dv}{Dt}$ and $\dfrac{Dw}{Dt}$.

Substituting in (1) we obtain, as the equations of motion,

$$\left.\begin{aligned}\frac{1}{\rho}\frac{dp}{dx}&=X-\frac{du}{dt}-u\frac{du}{dx}-v\frac{du}{dy}-w\frac{du}{dz},\\ \frac{1}{\rho}\frac{dp}{dy}&=Y-\frac{dv}{dt}-u\frac{dv}{dx}-v\frac{dv}{dy}-w\frac{dv}{dz},\\ \frac{1}{\rho}\frac{dp}{dz}&=Z-\frac{dw}{dt}-u\frac{dw}{dx}-v\frac{dw}{dy}-w\frac{dw}{dz}.\end{aligned}\right\}\quad\ldots\ldots\ldots(2).$$

If the fluid be elastic, and if the temperature be supposed constant, we have also the equation

$$p = k\rho.$$

It is important to observe carefully the meanings of the symbols employed in these equations; the quantities $\frac{Du}{Dt}$, $\frac{Dv}{Dt}$, $\frac{Dw}{Dt}$ in (1), are the rates of variation of the velocities, parallel to the axes, which, at the time t, represent the motion of the particular element m of fluid; and in both (1) and (2) the differential coefficients $\frac{dp}{dx}$, $\frac{dp}{dy}$, $\frac{dp}{dz}$, are taken on the supposition that x, y, and z are independent variables.

117. Let a, b, c, be the co-ordinates which define the initial position of the element m, and let

$$x = a + \xi, \qquad y = b + \eta, \qquad z = c + \zeta;$$

then ξ, η, ζ, are the displacements, parallel to the axes, of the particle m during the time t, and, x, y, z, being functions of a, b, c, and t, that is, of $x - \xi$, $y - \eta$, $z - \zeta$, and t, it follows that ξ, η, ζ, are functions of x, y, z, and t, and therefore

$$\frac{D\xi}{Dt} = \frac{d\xi}{dt} + \frac{d\xi}{dx}\frac{Dx}{Dt} + \frac{d\xi}{dy}\frac{Dy}{Dt} + \frac{d\xi}{dz}\frac{Dz}{Dt}.$$

But $\qquad \frac{D\xi}{Dt} = u,$

$$\left.\begin{aligned} \therefore\quad & \frac{d\xi}{dt} + u\frac{d\xi}{dx} + v\frac{d\xi}{dy} + w\frac{d\xi}{dz} = u, \\ \text{and similarly,}\quad & \frac{d\eta}{dt} + u\frac{d\eta}{dx} + v\frac{d\eta}{dy} + w\frac{d\eta}{dz} = v, \\ & \frac{d\zeta}{dt} + u\frac{d\zeta}{dx} + v\frac{d\zeta}{dy} + w\frac{d\zeta}{dz} = w. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (3).$$

These equations will determine ξ, η, ζ, when u, v, w have been determined as functions of x, y, z, and t.

The Equation of Continuity.

118. The equations obtained in Art. (116), are not sufficient for the purpose of determining the motion, and another equation can be obtained from the consideration that the fluid, although its external form may change, or the density about any point within it vary during the motion, must be, in general, a *continuous* mass.

We shall express this continuity of the fluid by assuming that any small parallelopiped, fixed in space within the fluid, remains full during the motion which takes place in any small interval of time.

Let x, y, z, be the co-ordinates of one angular point, P, and $x+\alpha$, $y+\beta$, $z+\gamma$, of the opposite angular point of the parallelopiped.

Then, if ρ be the density, and u the velocity parallel to x, at the point P, the quantity of fluid which enters the parallelopiped at the face $\beta\gamma$, containing P, will be

$$\rho u\beta\gamma\delta t,$$

in the time δt, and therefore the quantity which, during the same time, flows out at the opposite face, will be

$$\left(\rho u + \frac{d \,.\, \rho u}{dx}\alpha\right)\beta\gamma\delta t.$$

Hence the loss of fluid in consequence of the motion parallel to x, is

$$\frac{d(\rho u)}{dx}\alpha\beta\gamma\delta t.$$

Similarly the quantities lost in consequence of the other motions, are

$$\frac{d(\rho v)}{dy}\alpha\beta\gamma\delta t, \text{ and } \frac{d(\rho w)}{dz}\alpha\beta\gamma\delta t,$$

and the total loss is,

$$\left\{\frac{d(\rho u)}{dx} + \frac{d(\rho v)}{dy} + \frac{d(\rho w)}{dz}\right\}\alpha\beta\gamma\delta t;$$

but the increase in the quantity of fluid in the time δt is given by the expression $\frac{d\rho}{dt}\delta t\,.\,\alpha\beta\gamma$, that is, the loss is

$$-\frac{d\rho}{dt}\alpha\beta\gamma\delta t,$$

and therefore, equating these expressions,

$$\frac{d\rho}{dt}+\frac{d(\rho u)}{dx}+\frac{d(\rho v)}{dy}+\frac{d(\rho w)}{dz}=0 \quad\text{..............} \quad (4).$$

This equation may also be obtained from the consideration that any particular elementary portion of the fluid remains a continuous mass during its subsequent motion. Thus, if x, y, z, be the co-ordinates, at the time t, of a particle of the fluid, the equation is obtained by equating to zero the variation, in the time δt, of the element $\rho\delta x\delta y\delta z$.

119. If the fluid be homogeneous and incompressible, ρ is constant, and the equation becomes

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0.$$

This last equation is also true, if the fluid be heterogeneous, provided it be incompressible; for the density of a particle in motion will be invariable; and therefore the variation of ρ, considered as a function of x, y, z, and t, will be zero, if we take

$$\delta x = u\delta t, \quad \delta y = v\delta t, \quad \text{and } \delta z = w\delta t.$$

Hence

$$\frac{d\rho}{dt}+u\frac{d\rho}{dx}+v\frac{d\rho}{dy}+w\frac{d\rho}{dz}=0,$$

and, subtracting this from the general equation of continuity, we get

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0.$$

120. If the forces be such that $Xdx+Ydy+Zdz$ is the complete differential dR of some function R, and if the motion be of such a nature that $udx+vdy+wdz$ is a complete differential $d\phi$, the several equations can be reduced to a more manageable form.

In this case $\frac{d\phi}{dx} = u$, $\frac{d\phi}{dy} = v$, and $\frac{d\phi}{dz} = w$;

and $\therefore$ $\frac{Du}{Dt} = \frac{d^2\phi}{dxdt} + \frac{d\phi}{dx} \cdot \frac{d^2\phi}{dx^2} + \frac{d\phi}{dy} \cdot \frac{d^2\phi}{dxdy} + \frac{d\phi}{dz} \cdot \frac{d^2\phi}{dxdz}$,

$$\frac{Dv}{Dt} = \frac{d^2\phi}{dydt} + \frac{d\phi}{dx} \cdot \frac{d^2\phi}{dxdy} + \frac{d\phi}{dy} \cdot \frac{d^2\phi}{dy^2} + \frac{d\phi}{dz} \cdot \frac{d^2\phi}{dydz},$$

$$\frac{Dw}{Dt} = \frac{d^2\phi}{dzdt} + \frac{d\phi}{dx} \cdot \frac{d^2\phi}{dxdz} + \frac{d\phi}{dy} \cdot \frac{d^2\phi}{dydz} + \frac{d\phi}{dz} \cdot \frac{d^2\phi}{dz^2}.$$

From the equations (1) we have

$$\frac{1}{\rho} dp = \left(X - \frac{Du}{Dt}\right) dx + \left(Y - \frac{Dv}{Dt}\right) dy + \left(Z - \frac{Dw}{Dt}\right) dz;$$

and $\therefore$ $\frac{1}{\rho} dp = dR - d \cdot \frac{d\phi}{dt} - \frac{1}{2} d \cdot \left\{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2\right\}$,

or $$\frac{1}{\rho} dp = dR - d \cdot \frac{d\phi}{dt} - \frac{1}{2} d(U^2) \quad \text{.................(5)},$$

U being the resultant velocity at the point x, y, z.

Hence, if the fluid be inelastic and homogeneous,

$$\frac{p}{\rho} = R - \frac{d\phi}{dt} - \frac{1}{2} U^2,$$

and the equation of continuity is

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0.$$

If the fluid be elastic, $p = k\rho$, and we obtain

$$k \log p = R - \frac{d\phi}{dt} - \frac{1}{2} U^2.$$

In each integration an arbitrary function of t must be introduced, but it is unnecessary to insert such a function in the equation, as it may be supposed to be contained in $\frac{d\phi}{dt}$.

121. Taking s an arc of the line of motion passing through

the point x, y, z, it is evident that $m\frac{dR}{ds}$ is the force on the particle m in the direction of its motion; for, since

$$X=\frac{dR}{dx},\quad Y=\frac{dR}{dy},\ \text{and}\ Z=\frac{dR}{dz},$$

$$\frac{dR}{ds}=X\frac{dx}{ds}+Y\frac{dy}{ds}+Z\frac{dz}{ds}.$$

Also the velocity $U=u\frac{dx}{ds}+v\frac{dy}{ds}+w\frac{dz}{ds}=\frac{d\phi}{ds}$;

and therefore, if mS be the tangential force on m, the equation (5) may be written

$$\frac{1}{\rho}\frac{dp}{ds}=S-\frac{dU}{dt}-U\frac{dU}{ds}.$$

This equation may be obtained more briefly as follows. Taking p as the pressure at any point of a fluid at rest, and measuring s in any direction, we have

$$\frac{dp}{ds}=\rho S,$$

where mS measures the force on m in the direction of s.

In the case of the fluid in motion, measure s in the direction of the line of motion, and let U be the velocity; then U is a function of s and t, and

$$\frac{DU}{Dt}=\frac{dU}{dt}+U\frac{dU}{ds}.$$

Hence by D'Alembert's principle,

$$\frac{1}{\rho}\frac{dp}{ds}=S-\frac{dU}{dt}-U\frac{dU}{ds}\ \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

122. Cases of motion are of course conceivable in which $udx+vdy+wdz$ is not a complete differential, and in such cases we must employ the equations (1) in order to determine the pressure at any point.

For instance, if a fluid revolve uniformly, without change of form or relative displacement, about a fixed axis,

$$udx+vdy+wdz,$$

is not a complete differential. Thus, taking the fixed axis as the axis of z,

$$u = -\omega y, \quad v = \omega x, \quad \text{and } w = 0;$$

$$\therefore\ udx + vdy + wdz = \omega\,(xdy - ydx),$$

an expression which clearly is not an exact differential.

Recurring to the equations (1), we have, for this case,

$$\frac{1}{\rho}\frac{dp}{dx} = X + \omega^2 x, \quad \frac{1}{\rho}\frac{dp}{dy} = Y + \omega^2 y, \quad \frac{1}{\rho}\frac{dp}{dz} = Z;$$

and therefore,

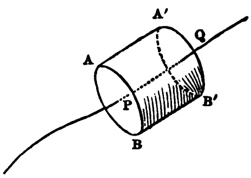
$$\frac{1}{\rho}\,dp = Xdx + Ydy + Zdz + \omega^2\,(xdx + ydy),$$

as in Art. (31).

It can be shewn however, that if, at any time during the motion, $udx + vdy + wdz$ is a perfect differential, it is so always; a proof of this will be given hereafter.

123. Another form of the *equation of continuity*, which is sometimes useful, may be obtained as follows.

Let $PQ = \delta s$ be an arc of the line of motion passing through a point Q; and let AB be a small area normal to the arc, such that all the particles of fluid crossing it may be considered as moving perpendicularly to it.



Let AA', BB', &c. be small arcs of the lines of motion through the bounding points of AB, and $A'B'$ the normal section through Q of the surface formed by these lines of motion.

Take ρ as the density of the fluid in PQ at the time t, κ the area of AB, and v the velocity at P; then the quantity of fluid which enters at AB during the time δt

$$= \kappa\rho v\delta t,$$

and that which flows out at $A'B'$

$$= \kappa\rho v\delta t + \frac{d}{ds}\,(\kappa\rho v\delta t)\,.\,\delta s.$$

The excess of the former over the latter of these two expressions is the whole increase of the fluid in PQ during the time δt, and is

$$-\frac{d}{ds}(\kappa\rho v)\,\delta t\delta s:$$

but the mass of fluid at the time t being $\kappa\rho\delta s$, the increase in the time δt is also expressed by

$$\frac{d}{dt}(\kappa\rho\delta s)\,\delta t, \quad \text{or} \quad \frac{d}{dt}(\kappa\rho)\,\delta s\delta t,$$

and therefore

$$\frac{d}{dt}(\kappa\rho)+\frac{d}{ds}(\kappa\rho v)=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

From the way in which this equation has been obtained, it will be seen that allowance is made for the expansion of the element which may in certain cases take place, and it is only in this way that κ can be an explicit function of time. The small section AB may be taken arbitrarily, but the section $A'B'$ will depend, not only on the arc PQ, but also on the directions of the lines of motion passing through the bounding curve of AB; the variation of κ may therefore depend on the time explicitly, since these lines of motion may vary with the time.

124. The form (7) may also be obtained by considering the motion of a small cylindrical portion PQ of fluid, and by expressing the condition that its mass should remain the same.

Let PQ be the position of the element at the time t,

$P'Q'$ at the time $t+dt$,

v the velocity of P, and therefore $v+\dfrac{dv}{ds}ds$ of Q, if $PQ=\delta s$;

then $PP'=vdt$, and $QQ'=(v+\dfrac{dv}{ds}\delta s)\,\delta t$,

and therefore $P'Q' = \delta s\,(1 + \frac{dv}{ds}\,\delta t)$.

If κ be the section of the cylinder PQ,

κ is a function of s and t, where s also depends upon t,

and therefore the section of $P'Q' = \kappa + \frac{d\kappa}{dt}\,dt + \frac{d\kappa}{ds}\,.\,vdt$.

Hence the mass of $P'Q'$

$$= (\rho + \frac{d\rho}{dt}\,dt + \frac{d\rho}{dr}\,vdt)\,(\kappa + \frac{d\kappa}{dt}\,dt + \frac{d\kappa}{dr}\,vdt)\,P'Q'$$

$$= \text{mass of } PQ = \rho\kappa\,.\,PQ = \rho\kappa\,.\,\delta s,$$

and we thence obtain $\frac{d}{dt}(\kappa\rho) + \frac{d}{ds}(\kappa\rho v) = 0$.

This form of the equation of continuity will be found available in cases in which it may be convenient to employ the equation of motion, (6), obtained in Art. (121).

The Bounding Surface.

125. In whatever manner a fluid mass be in motion, the particles of fluid which at any time happen to be in the surface can have no motion across it; in other words, the fluid particles either have no motion relative to the surface, or their relative motion is wholly tangential.

It is not to be supposed that particles once in the surface, are always in the surface, for the motion of a fluid particle, relatively, may be in a curve touching the surface, or it may approach with a relative velocity continually diminishing and vanishing at the surface, and may then retreat within the fluid mass; all these cases will however be included in the condition that the relative motion is tangential to the surface.

Let $$F(\xi, \eta, \zeta, t) = 0,$$

be the equation to the bounding surface at the time t, and let x, y, z, be the co-ordinates of a fluid particle in the surface; then

$$F(x, y, z, t) = 0.$$

At the time $t + \delta t$, the co-ordinates of the particle are

$$x + u\delta t, \quad y + v\delta t, \quad z + w\delta t,$$

and, since the relative motion is tangential, these quantities will differ by small quantities of the second order from the co-ordinates of *some* point in the surface

$$F(\xi, \eta, \zeta, t + \delta t) = 0;$$

and, therefore, substituting for ξ, η, ζ, and neglecting such quantities,

$$F(x + u\delta t, \ y + v\delta t, \ z + w\delta t, \ t + \delta t) = 0,$$

from which we obtain in the limit

$$F'(t) + uF'(x) + vF'(y) + wF'(z) = 0,$$

the differential equation to the surface.

126. If the fluid be incompressible, and if ϖ be the external pressure upon its surface, and p the pressure at the surface, we shall have

$$p = \varpi,$$

and, therefore, at all points of the free surface,

$$\frac{dp}{dt} + u\frac{dp}{dx} + v\frac{dp}{dy} + w\frac{dp}{dz} = \frac{d\varpi}{dt};$$

u, v, and w being the velocities of the point x, y, z.

CHAPTER IX.

STEADY MOTION AND PARALLEL SECTIONS.

127. WHEN the motion of a fluid is such that the velocity of a fluid particle is a function of the co-ordinates x, y, z only, and does not involve the time explicitly, that is, when the velocities of the particles of fluid which pass in succession through a given point are always the same, the motion is characterised as *steady motion*.

On this hypothesis, or, in other words, in the cases for which such a motion is possible, the expressions $\frac{du}{dt}, \frac{dv}{dt}, \frac{dw}{dt}$, do not appear in the original equation, and $\frac{d\phi}{dt}$ will therefore not appear in the final equation, which is, in consequence,

$$\frac{1}{\rho}\frac{dp}{ds} = S - v\frac{dv}{ds},$$

employing v to represent the velocity.

As an instance of steady motion, consider the case of a vessel kept constantly full, and having a horizontal orifice in its base, from which the fluid issues at an uniform rate. The vessel may be supposed to be in the form of a surface of revolution, and to have its base horizontal.

Gravity being the only force in action, $S = g\frac{dz}{ds}$, if z be measured vertically downwards, and

$$\frac{1}{\rho}\frac{dp}{ds} = g\frac{dz}{ds} - v\frac{dv}{ds};$$

$$\therefore\ \frac{p}{\rho} = gz - \tfrac{1}{2}v^2 + C.$$

Let U be the velocity at the surface, and u at the orifice;

then, taking h as the depth of the orifice below the surface, and Π as the atmospheric pressure,

$$\frac{\Pi}{\rho} = C - \tfrac{1}{2}U^2,$$

$$\frac{\Pi}{\rho} = gh + C - \tfrac{1}{2}u^2,$$

$$\therefore\ u^2 = U^2 + 2gh.$$

But, if A be the area of the surface, and K of the orifice, and if the motions of all the issuing particles be supposed perpendicular to the plane of the orifice,

$$AU = Ku,$$

since the quantity of fluid poured in at the surface in any time is equal to the quantity which passes through the orifice in the same time;

$$\therefore\ U = \sqrt{(2gh)}\,\frac{K}{\sqrt{(A^2 - K^2)}},$$

$$\text{and } u = \sqrt{(2gh)}\,\frac{A}{\sqrt{(A^2 - K^2)}}.$$

If the orifice be very small, the ratio $\dfrac{K}{A}$ may be neglected, and, approximately, $u = \sqrt{(2gh)}$.

128. In any case of steady motion, if gravity be the only force in action, we have

$$\frac{p}{\rho} = gz - \tfrac{1}{2}v^2 + C.$$

Suppose the orifice not in the base of the vessel, and so small that the velocities of all the particles passing through it are sensibly the same; we then have, as in the previous case,

$$u^2 = U^2 + 2gh,\ \ AU = Ku,$$

and approximately, $u = \sqrt{(2gh)}$.

If the vessel be not kept constantly full, the motion will not be steady, but when the orifice is very small, it may be taken as being approximately steady, and the expression $\sqrt{(2gh)}$ may be employed as the velocity of the issuing fluid.

Looking upon the issuing fluid as a series of particles in motion under the action of gravity, every particle moves in a parabolic path, and the issuing fluid takes the form of a parabolic arc. Moreover since the velocity at the orifice is, approximately, that due to the height h, the directrix of the parabola is approximately coincident with the surface of the fluid.

129. PROP. *To find the time in which a given quantity of fluid will flow through a small orifice.*

At the time t, let x be the height of the surface above the orifice, and X its area.

Then, approximately,

$$\text{velocity at the orifice} = \sqrt{(2gx)}:$$

$$\text{but } -\frac{dx}{dt} \text{ is the velocity of the surface,}$$

$$\therefore\ -X\frac{dx}{dt} = \kappa\sqrt{(2gx)},$$

$$\text{or } \frac{dt}{dx} = -\frac{X}{\kappa\sqrt{(2gx)}}.$$

X being a known function of x, this equation gives t in terms of x, and therefore x in terms of t.

It will be seen hereafter that, in certain cases, particularly when the containing vessel is formed of a thin substance, a considerable modification of the value of k, employed in the preceding process, is requisite, in order to obtain results in approximate accordance with observations.

130. Ex. 1. *A hollow cone, having its axis vertical, is filled with water; required to find the time in which it will be emptied through a small aperture at its vertex.*

In this case, $X = \pi x^2 \tan^2 \alpha$, taking 2α as the vertical angle;

$$\therefore\ \frac{dt}{dx} = -\frac{\pi\tan^2\alpha}{k\sqrt{(2g)}}\,x^{\frac{3}{2}},$$

and, if h be the height of the cone, the time (t) in which it will be emptied is

$$-\int_h^0 \frac{\pi \tan^2 \alpha}{\kappa \sqrt{(2g)}} x^{\frac{3}{2}} dx, \quad \text{or} \quad \frac{1}{5} \frac{\pi h^2 \tan^2 \alpha}{\kappa} \sqrt{\left(\frac{2h}{g}\right)}.$$

If the cone had been kept constantly full, the velocity at the orifice would have been always $\sqrt{(2gh)}$, and the same quantity of liquid would have flowed out in a time τ, such that

$$\tau\kappa \sqrt{(2gh)} = \tfrac{1}{3}\pi h^3 \tan^2 \alpha;$$

hence we obtain $t : \tau :: 6 : 5$.

Ex. 2. *A vessel, in the form of a surface of revolution, has a small aperture at its lowest point; determine its form so that the surface of water, contained in it, may descend uniformly.*

We must have $\frac{dx}{dt}$ constant, and therefore $\frac{X}{\sqrt{x}}$ constant; but, if $y = f(x)$ be the generating curve, $X = \pi y^2$, and therefore $\frac{y^2}{\sqrt{x}}$ is constant: hence the generating curve is one of the class

$$y^4 = a^3 x,$$

the velocity of descent being determined by the value of a.

This example contains the theory of the Clepsydra or water-clock.

The Hypothesis of Parallel Sections.

131. Suppose the interior of a vessel to be a surface of revolution, the axis of which is vertical; and suppose moreover that the inclination to the vertical of the generating curve is always small, and does not change rapidly.

If such a vessel contain fluid, which is allowed to flow out through a horizontal aperture in its base, it is evident that the fluid particles will move in directions nearly vertical, and the velocities of all particles in the same horizontal plane will be very nearly the same. The discussion of the real motions in such a case would be excessively complicated, but an approxi-

mate solution may be obtained by means of the hypothesis, that the successive horizontal laminæ of fluid descend vertically, and replace each other in succession, that is, that the motions of all the fluid particles in a horizontal plane are the same, and all vertical.

This is the hypothesis of parallel sections, and it is clearly equivalent to the neglecting of all horizontal motions, and of the changes which take place in the component particles of the descending laminæ of fluid.

If the orifice be much less than the horizontal base of the vessel, the motions of the particles near the base cannot be all vertical, and the same, in the same horizontal plane; the hypothesis therefore will not even approximately hold good. In order however to obtain a solution of the question, the hypothesis will be made throughout, and a large allowance must therefore be made for the probable error arising from this cause.

Under this head we shall discuss the following problems.

132. I. *A vase in the form of a surface of revolution, and having a finite horizontal aperture in its base, is kept constantly full; required to determine the rate at which fluid must be poured in.*

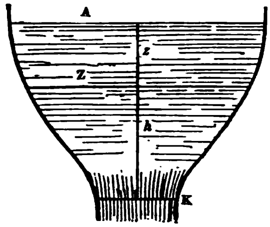
Let A be the area of the top of the vase, K of the aperture, and h the depth of the vase.

At a depth z below the surface, where Z is the area of the horizontal section, let v be the velocity at the time t, and, at the same time, let U be the velocity at the surface and u at the aperture.

Then, the fluid being supposed incompressible, the same quantity must pass through any horizontal section in the same element of time δt;

$$\therefore\ U\delta t\,.\,A = u\delta t K = v\delta t\,.\,Z,$$

$$\text{or } AU = Ku = Zv.$$



These conditions, it will be observed, express the *continuity* of the fluid.

The only force acting on the fluid is gravity;

$$\therefore\ S = g,$$

and the equation of motion is

$$\frac{1}{\rho}\frac{dp}{dz} = g - \frac{dv}{dt} - v\frac{dv}{dz}.$$

Now U and u are functions of t, but are independent of z: Z, being the area of the section for which the velocity is v, is a function of z, and is therefore only implicitly a function of t.

Hence the equation $v = \dfrac{AU}{Z}$,

$$\text{gives } \frac{dv}{dt} = \frac{A}{Z}\frac{dU}{dt};$$

$$\therefore\ \frac{1}{\rho}\frac{dp}{dz} = g - \frac{A}{Z}\frac{dU}{dt} - v\frac{dv}{dz},$$

$$\text{and } \frac{p}{\rho} = C + gz - A\frac{dU}{dt}\int\frac{dz}{Z} - \tfrac{1}{2}v^2,$$

where C may be a function of t.

Let Π be the pressure at the surface;

then, when $z = 0$, $p = \Pi$, $v = U$,

$$\text{and } \frac{p - \Pi}{\rho} = gz - A\frac{dU}{dt}\int_0^z \frac{dz}{Z} - \tfrac{1}{2}(v^2 - U^2)$$

$$= gz - A\frac{dU}{dt}\int_0^z \frac{dz}{Z} - \tfrac{1}{2}U^2\left(\frac{A^2}{Z^2} - 1\right).$$

Let Π' be the pressure at the orifice;

putting $z = h$, and therefore $Z = K$,

$$\frac{\Pi' - \Pi}{\rho} = gh - A\frac{dU}{dt}\int_0^h \frac{dz}{Z} - \tfrac{1}{2}U^2\left(\frac{A^2}{K^2} - 1\right).$$

If the vase be in air Π' and Π will be sensibly the same, and,

assuming this to be the case, we have, for the determination of U, the equation

$$A\int_0^h \frac{dz}{Z}\,\frac{dU}{dt} = gh - \frac{1}{2}\left(\frac{A^2}{K^2} - 1\right)U^2.$$

Let $A\int_0^h \frac{dz}{Z} = a$, and $\frac{A^2}{K^2} - 1 = 2m$;

$$\text{then } a\frac{dU}{dt} = gh - mU^2,$$

$$dt = \frac{a\,dU}{m\left(\frac{gh}{m} - U^2\right)};$$

$$\therefore \frac{m}{a}\,t = \frac{1}{2}\sqrt{\left(\frac{m}{gh}\right)}\log\frac{\sqrt{\left(\frac{gh}{m}\right)} + U}{\sqrt{\left(\frac{gh}{m}\right)} - U} + C',$$

$$\text{or, } \frac{\sqrt{\left(\frac{gh}{m}\right)} + U}{\sqrt{\left(\frac{gh}{m}\right)} - U} = C\epsilon^{\frac{2\sqrt{(ghm)}}{a}t};$$

$$\therefore U = \sqrt{\left(\frac{gh}{m}\right)}\cdot\frac{C\epsilon^{\alpha t} - 1}{C\epsilon^{\alpha t} + 1}, \text{ if } 2\sqrt{(ghm)} = a\alpha$$

$$= \sqrt{\left(\frac{gh}{m}\right)}\cdot\frac{C - \epsilon^{-\alpha t}}{C + \epsilon^{-\alpha t}}.$$

Suppose that initially the vase was just filled, and the fluid then allowed to escape at the orifice, the vase being kept full by pouring in fluid above; then initially $U = 0$;

$$\therefore U = \sqrt{\left(\frac{gh}{m}\right)}\,\frac{1 - \epsilon^{-\frac{2\sqrt{(ghm)}}{a}t}}{1 + \epsilon^{-\frac{2\sqrt{(ghm)}}{a}t}};$$

this equation determines the rate at which fluid is being poured in at the time t.

The quantity which has been poured in from the beginning of the motion to the time t'

$$= \int_0^{t'} UA\, dt$$

$$= A\sqrt{\left(\frac{gh}{m}\right)}\int_0^{t'} \frac{\epsilon^{\alpha t} - 1}{\epsilon^{\alpha t} + 1}\, dt$$

$$= A\sqrt{\left(\frac{gh}{m}\right)}\int_0^{t'} \left\{1 - \frac{2\epsilon^{-\alpha t}}{1+\epsilon^{-\alpha t}}\right\} dt$$

$$= A\sqrt{\left(\frac{gh}{m}\right)}\left\{t' + \frac{2}{\alpha}\log\frac{1+\epsilon^{-\alpha t'}}{2}\right\}.$$

If the motion be continued for a long period of time, we observe that U appoximates to a 'terminal velocity' $\sqrt{\left(\frac{gh}{m}\right)}$,

$$\text{or, } \sqrt{\left(2gh \cdot \frac{K^2}{A^2 - K^2}\right)}.$$

$$\text{Also, } u = \frac{AU}{K} = \sqrt{\left(2gh\ \frac{A^2}{A^2 - K^2}\right)}\ \frac{1 - \epsilon^{-\alpha t}}{1 + \epsilon^{-\alpha t}},$$

and approximates to a terminal value,

$$\sqrt{\left(2gh\ \frac{A^2}{A^2 - K^2}\right)}.$$

If K is small compared with A, these are, approximately,

$$\frac{K}{A}\sqrt{(2gh)}, \text{ and } \sqrt{(2gh)},$$

results which might have been anticipated, for it is clear that ultimately the motion will become 'steady.'

133. II. *A vase, having a horizontal aperture in its base, contains liquid, which is allowed to flow out through the orifice; required to determine the motion.*

At the time t,

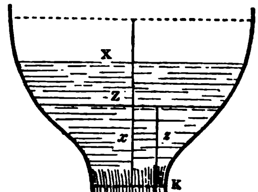
let x be the height of the surface above the orifice,

X the area of the section at the surface;

a, A, the initial values of x and X,

Z the area of the section at a height z about the orifice,

U the velocity at the surface, u at the orifice, and v at the height z.



The equation of motion is

$$\frac{1}{\rho}\frac{dp}{dz} = -g - \frac{dv}{dt} - v\frac{dv}{dz};$$

also $Ku = XU = Zv$,

where u is a function of t or x,

X of x, U of x and t, Z of z,

v of z and t, and x of t;

$$\therefore \frac{dv}{dt} = \frac{K}{Z}\frac{du}{dt},$$

$$\frac{1}{\rho}\frac{dp}{dz} = -g - \frac{K}{Z}\frac{du}{dt} - v\frac{dv}{dz},$$

$$\text{and } \frac{p}{\rho} = C - gz - K\frac{du}{dt}\int\frac{dz}{Z} - \frac{1}{2}v^2,$$

$$= C - gz - K\frac{du}{dt}\int\frac{dz}{Z} - \frac{1}{2}\frac{K^2u^2}{Z^2}.$$

At the time t, when $z = x$,

$$p = \Pi, \text{ and } Z = X;$$

and, when $z = 0$, $p = \Pi$, and $Z = K$;

$$\therefore 0 = -gx - K\frac{du}{dt}\int_0^x\frac{dz}{Z} - \frac{1}{2}K^2u^2\left(\frac{1}{X^2} - \frac{1}{K^2}\right).$$

$$\text{Now } \frac{du}{dt} = \frac{du}{dx}\frac{dx}{dt} = -U\frac{du}{dx},$$

U being measured downwards and x upwards,

$$= -\frac{Ku}{X}\frac{du}{dx};$$

$$\therefore \frac{K^2u}{X}\frac{du}{dx}\cdot\int_0^x\frac{dz}{Z} - \frac{1}{2}\left(\frac{K^2}{X^2} - 1\right)u^2 - gx = 0;$$

$$\text{or, } \frac{K^2}{X}\int_0^x \frac{dz}{Z}\frac{d.u^2}{dx} - \left(\frac{K^2}{X^2} - 1\right)u^2 - 2gx = 0,$$

a linear equation which determines u, and therefore U, in terms of x;

and, from the equation

$$\frac{dx}{dt} = -U,$$

we can obtain t in terms of x, and therefore x in terms of t.

The quantity of fluid which has escaped in the time t from the beginning of the motion is the volume of the vase between the heights x and a, that is,

$$\int_x^a Zdz.$$

It may also be expressed as the quantity which has flowed through the orifice in the time t, which

$$= \int_0^t Kudt.$$

Consequently

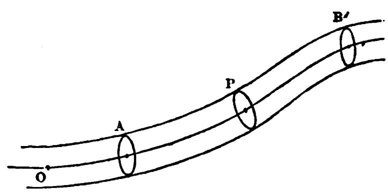
$$\int_x^a Zdz = K\int_0^t udt,$$

observing that x is a function of t.

As before, if K is very small compared with the values of Z, $\frac{K^2}{X^2}$ and $\frac{K^2}{X}\int_0^x \frac{dz}{Z}$ may be neglected, and, as a rough approximation we have, $u^2 = 2gx$.

134. III. *The motion of an incompressible fluid in a tube of small section.*

We shall suppose that the particles of fluid in any normal section move perpendicularly to the section, and that the volume of fluid is given.



Let O be a fixed point in the axis of the tube, AB the fluid in motion, Z the area of the section at a point P in the fluid, A, A', the sections at the extremities.

Take $OA = a$, $OB = a'$, $OP = s$, and let S be the acceleration at P in direction of the axis;

$$\therefore \frac{1}{\rho}\frac{dp}{ds} = S - \frac{dv}{dt} - v\frac{dv}{ds},$$

where v is the velocity at P.

If u, u' be the velocities at A and B,

$$Au = A'u' = Zv,$$

where A is a function of a, A' of a' and Z of s;

$$\therefore \frac{dv}{dt} = \frac{A}{Z}\frac{du}{dt}, \text{ and } \frac{dv}{ds} = -\frac{Au}{Z^2}\frac{dZ}{ds},$$

$$\frac{1}{\rho}\frac{dp}{ds} = S - \frac{A}{Z}\frac{du}{dt} + \frac{A^2u^2}{Z^3}\frac{dZ}{ds},$$

and, integrating with regard to s,

$$\frac{p}{\rho} = C + \int Sds - A\frac{du}{dt}\int\frac{ds}{Z} - \frac{1}{2}\frac{A^2u^2}{Z^2}, \text{.........(1).}$$

where C may be a function of the time.

Let the pressures at A and B be equal, then

$$o = \int_a^{a'} Sds - A\frac{du}{dt}\int_a^{a'}\frac{ds}{Z} - \frac{1}{2}A^2u^2\left\{\frac{1}{A'^2} - \frac{1}{A^2}\right\}\text{... (2).}$$

Also, since the volume of fluid is given (V suppose),

$$\int_a^{a'} Zds = V,$$

which gives a' in terms of a, and therefore A' as well as A in terms of a.

Hence the equation (2) may be written in the form

$$\frac{du}{dt} + \phi(a)u^2 + \psi(a) = 0,$$

or, since $$u = \frac{da}{dt},$$

$$\frac{d^2\alpha}{dt^2}+\phi(\alpha)\left(\frac{d\alpha}{dt}\right)^2+\psi(\alpha)=0,$$

a differential equation from which α may in some cases be found in terms of t, and thence α', A, A', and p, from (1).

If gravity be the only force, $S=g\frac{dz}{ds}$, measuring z downwards, and

$$\int_\alpha^{\alpha'} Sds=\int_\gamma^{\gamma'} gdz=g(\gamma'-\gamma),$$

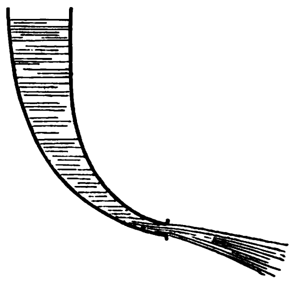
γ and γ' being the values of z at A and B.

The equation (2) may also be written

$$o=\int_\alpha^{\alpha'} Sds-A'\frac{du'}{dt}\int_\alpha^{\alpha'}\frac{ds}{Z}-\frac{1}{2}A'^2u'^2\left(\frac{1}{A'^2}-\frac{1}{A^2}\right),$$

and, if A' be small compared with A, the second and third terms may be neglected, and an approximate equation obtained for u'^2, i.e.

$$u'^2=2g(\gamma'-\gamma).$$



Suppose the tube, as in the figure, to end in an orifice from which the fluid issues. If we take the origin at the orifice, we shall have $\alpha'=0$, $\gamma'=0$, A' constant, and γ negative, and, as before, the velocity at the lowest point of the tube will be approximately the velocity due to the depth below the surface.

135. *The motion of an incompressible fluid in an uniform tube of small section.*

In this case v is the same at all points of the tube, and therefore $\frac{dv}{ds}=0$, and the equation of motion is

$$\frac{1}{\rho}\frac{dp}{ds}=S-\frac{dv}{dt}.$$

Let gravity be the only force in action, and measure z vertically upwards;

$$\therefore\ S=-g\frac{dz}{ds},$$

and $$\frac{p}{\rho} = C - gz - \frac{dv}{dt} s.$$

Taking α and α' as the extreme values of s, γ and γ' of z, we obtain

$$0 = g(\gamma - \gamma') + \frac{dv}{dt}(\alpha - \alpha'),$$

or, if l be the length of the filament of fluid,

$$l \frac{dv}{dt} = -g(\gamma - \gamma'),$$

observing that $$\frac{dv}{dt} = \frac{d\alpha}{dt} = \frac{d\alpha'}{dt}.$$

Ex. *Liquid rests in a fine tube, the axis of which is a circle in a vertical plane; the fluid being slightly disturbed, required the time of a small oscillation.*

Let the filament of fluid subtend an angle 2α at the centre, and at the time t let θ be the angular distance from the vertical of the middle point of the filament.

Then $\gamma - \gamma' = a\{\cos(\alpha - \theta) - \cos(\alpha + \theta)\} = 2a \sin\alpha \sin\theta$,

$$l = 2a\alpha, \text{ and } \frac{dv}{dt} = a\frac{d^2\theta}{dt^2},$$

$$\therefore\ a\alpha \frac{d^2\theta}{dt^2} = -g \sin\alpha \sin\theta,$$

and, if the original displacement be small, the time of a small oscillation is

$$\pi \sqrt{\left(\frac{a\alpha}{g \sin\alpha}\right)}.$$

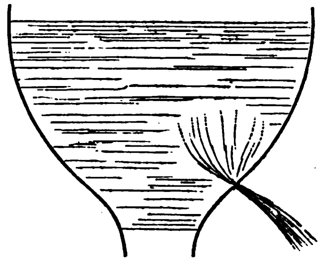
Small Orifices.

136. In each of the three preceding problems we have seen that, when the orifice is small, the velocity of efflux is approximately the velocity due to the depth of the orifice below the surface. This is in accordance with the result of Art. (127), in which it is assumed that the motion is *steady* in all cases in

which the orifice is small, and we are therefore now enabled, by observing the quantities neglected in making the approximations, to estimate the amount of error involved in taking the hypothesis of steady motion for such cases.

The case of a small orifice, not in a horizontal plane, may be illustrated by the third problem, Art. (134).

For it may be easily conceived that the fluid below the orifice will be almost entirely at rest, and that the issuing



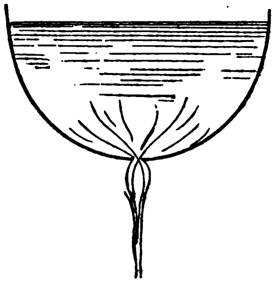
stream, or the central portion of it, will, before its efflux, have been flowing through the fluid in the vessel in a somewhat tubular form, so that its motion may be considered as the motion of a fluid in a tube, the section of which continually diminishes near the orifice; and therefore the result of the problem referred to may be applied to the confirmation of the result before obtained on the hypothesis of steady motion.

The contracted vein.

137. When fluid issues through a small orifice in the thin base of a vessel, it is observed that the issuing stream is not cylindrical, but, near the orifice, is contracted so that its sectional area is less than the area of the orifice. The stream then expands and afterwards, as it descends, again diminishes gradually in size.

The sudden diminution of the issuing stream forms what is called the 'contracted vein,' and is due to the oblique or nearly horizontal motions of the particles near the edges of the orifice just before their efflux.

The after contraction, which is gradual, is due to the law of continuity, which requires that the mean velocity of the particles



in any horizontal section of the issuing stream should vary inversely as the area of the section, and therefore that, as the velocity increases in the descent, the area of the section should diminish.

138. The discrepancy which exists between the results of theory and experiment is to a great extent accounted for by the contraction of the vein or filament of issuing fluid, and it is found moreover, as would be anticipated, that the amount of difference depends upon the nature of the orifice.

For instance, if the orifice be simply an opening in the side of the vessel, and if the side be very thin, the quantity of fluid which flows out in a given time is about $\frac{3}{5}$ths of the quantity given by the theory. Again, when the fluid issues through a cylindrical aperture of sensible length, formed by attaching to the orifice, externally, a small hollow cylinder, the ratio is found to be about $\frac{4}{5}$ths; but, if the cylinder be attached internally, the rate of efflux is about one half the theoretical rate*.

The *rate of efflux* depends upon the area of the orifice and the velocity of the issuing stream; it is shewn by experiment that the latter is, in general, not very different from the theoretical velocity, and the observed error in the rate of efflux is therefore to a great extent accounted for by the formation of the 'contracted vein.'

An account of experiments, made by Bossut and others, on the efflux of fluids through orifices of various kinds, is given in the Encyclopædia Metropolitana, *Hydrodynamics*, p. 207.

* Poisson, *Mécanique*, Art. 676.

139. In all the preceding investigations the containing vessel has been supposed to have the form of a surface of revolution, but they are evidently applicable to the case of a vessel of any form, the horizontal section of which does not change very rapidly, and the symbols employed (K, Z, &c.), being perfectly general, no correction is requisite for the application of the formulæ to such cases.

Motion of Elastic Fluids.

140. If elastic fluid move in a tube the section of which does not change rapidly in size, we may make use of the hypothesis of parallel sections as before.

Assuming the motions of all the particles in any one section to be sensibly in the same direction, parallel to the axis of the tube, and neglecting gravity, the action of which will not sensibly affect the pressure, the equation of motion is

$$\frac{1}{\rho}\frac{dp}{dx} = -\frac{dv}{dt} - v\frac{dv}{dx},$$

where v is the velocity in a section at a distance x from a fixed section.

The equation of continuity, depending on the hypothesis which neglects all motions but those perpendicular to the section, is determined as follows.

Let X be the area of the section, the velocity of the particles passing through which is v, and ρ the density about this section at the time t.

Then $\rho v X \delta t$ is the mass of fluid which flows across the section in the time δt;

$$\therefore\ \{\rho v X + \frac{d}{dx}(\rho v X)\,\delta x\}\,\delta t$$

is the quantity which flows across the section defined by the distance $x + \delta x$, and $-\frac{d}{dx}(\rho v X)\,\delta x\,\delta t$ is the increase of the quantity of fluid in the volume $X\delta x$ during the time δt, which is also given by the expression

$$X\delta x\,.\left(\frac{d\rho}{dt}\,\delta t\right);$$

$$\therefore\ X\frac{d\rho}{dt}+\frac{d}{dx}(\rho vX)=0$$

is the equation of continuity.

We have also, if the temperature remain constant,

$$p=k\rho,$$

and our equations become

$$\left.\begin{aligned}\frac{k}{p}\,\frac{dp}{dx}+\frac{dv}{dt}+v\frac{dv}{dx}=0,\\ X\frac{dp}{dt}+\frac{d}{dx}(pvX)=0.\end{aligned}\right\}$$

141. We shall not discuss the system of partial differential equations just obtained, but proceed to consider the particular case in which the motion is steady.

It may be supposed that the air is supplied from a large reservoir at a constant pressure, and we shall then have

$$\frac{dv}{dt}=0,\quad \frac{dp}{dt}=0,$$

and

$$\left.\begin{aligned}\therefore\ \frac{k}{p}\,\frac{dp}{dx}+v\frac{dv}{dx}=0,\\ \frac{d}{dx}(pvX)=0,\end{aligned}\right\}$$

or

$$\left.\begin{aligned}k\log p+\tfrac{1}{2}v^2=C\\ pvX=C'\end{aligned}\right\}.$$

Let x be measured from a plane in which the pressure is sensibly the constant pressure, Π', of the reservoir, and let A be the area of the section, and U the velocity of the particles in it.

Also let u be the velocity of efflux,

K the area of the orifice, and Π the pressure.

$$\therefore\ \tfrac{1}{2}(u^2-U^2)+k\log\frac{\Pi}{\Pi'}=0,$$

or

$$u^2=U^2+2k\log\frac{\Pi'}{\Pi},$$

and

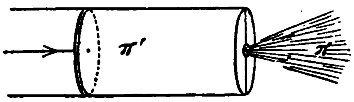
$$\Pi uK=\Pi' UA;$$

$$\therefore\ u^2\left(1-\frac{\Pi^2K^2}{\Pi'^2A^2}\right)=2k\log\frac{\Pi'}{\Pi}.$$

If U be very small, or K small compared with A, we have approximately

$$u^2 = 2k \log \frac{\Pi'}{\Pi}.$$

Suppose the air to be forced out of a cylinder through a small orifice by a piston moving slowly and exerting a constant pressure. The piston moving slowly with a velocity U, we may assume the motion as approximately steady;



$$\therefore\ k \log p + \tfrac{1}{2} v^2 = C;$$

and, as before, $$u^2 - U^2 = 2k \text{ by } \frac{\Pi'}{\Pi},$$

gives the velocity (u) of efflux.

142. From the equation of steady motion for elastic fluid, not under the action of any force,

$$k \frac{dp}{p} = -v \frac{dv}{ds},$$

we obtain $k \log p = C - \frac{1}{2} v^2$, or $p = \Pi \epsilon^{-\frac{v^2}{2k}}$,

Π being determined by knowing the pressure for a given velocity.

It follows therefore that p is diminished by an increase of velocity, a theoretical result which can be easily verified by experiment.

One form of the experiment is as follows. To one end of a straight tube let a plane disc be fitted which is capable of sliding on wires projecting from the end of the tube; if the disc be placed at a small distance from the end, and a person blow steadily into the tube, the disc will be *drawn* towards the tube, and, instead of being blown off the wires, will oscillate slightly about a position very near the end of the tube.

Or the experiment may be more simply performed by fastening a straw with sealing wax to a piece of card-board having a small hole in it. If a piece of paper be placed over the hole and

the experimenter blow through the straw, the paper will bend so as to allow the egress of the air, but will not be detached from the card.

The history of this experiment, and the variations which occur in practice for different sizes of the aperture and the disc, are given by Professor Willis, in the *Cambridge Philosophical Transactions*, Vol. III. Part I. The fact was first observed in some iron works in France, about 1826, where one of the forge-bellows opened in a flat wall, and it was found that a board presented to the blast was sucked up against the wall. An experiment was however devised by Hawksbee, in 1719, which is equally illustrative of the theory. Hawksbee's experiment simply consisted in passing a current of air through a small box, and he observed that the air contained in the box became considerably rarefied, a fact in accordance with the result that, neglecting changes of temperature, the pressure, and therefore also the density, is diminished by an increase of velocity*.

It may be here noticed, that an experiment, similar to the foregoing, was performed by M. Hachette, in 1826, with a stream of water, and with a similar result. The explanation is the same; that is, it appears from the equation of steady motion, for incompressible fluids, that the pressure diminishes with an increase of velocity.

143. In the preceding investigations on the motion of elastic fluids, the temperature has been considered uniform; if, however, the motion be very rapid, a sensible change of temperature takes place, and the results obtained must therefore, in such cases, be subject to considerable modification.

144. *A vessel, having a horizontal aperture in its base, is partially immersed in fluid of unlimited extent, and is kept constantly full of the same fluid; when the motion is steady, required to find the rate at which fluid is poured in.*

---

* I have since found that a theoretical explanation of the above experiment has been given by Professor Challis, *Camb. Phil. Trans.* Vol. I. Part III., who has also suggested other practical tests of the same theoretical result.

Let a be the height of the surface of the fluid in the vessel above the surface of the external fluid, and h the depth of the aperture below the upper surface.

Measuring z from the upper surface, the equation of steady motion is

$$p = g\rho z - \tfrac{1}{2}\rho v^2 + C;$$

but, when $z = 0$, $p = \Pi$, and, when $z = h$, $p = \Pi + g\rho(h - a)$, therefore, if u be the velocity at the upper surface and u' at the aperture,

$$u'^2 = u^2 + 2ga.$$

If K, K' be the areas of the surface and the aperture, $Ku = K'u'$, and the quantity poured in during the unit of time

$$= Ku = KK' \sqrt{\left(\frac{2ga}{K^2 + K'^2}\right)}.$$

145. If there be a *finite vertical orifice* in the side of a vessel containing liquid, the rate of efflux can be calculated, when the motion is steady, by supposing the orifice to consist of a number of very small orifices, and by determining the aggregate of the effluxes through all the orifices.

Thus, if u be the velocity at the surface, and v at an element of the orifice κ, the depth of which is z,

$$v^2 = u^2 + 2gz,$$

and taking K as the area of the surface,

$$Ku = \Sigma(\kappa v),$$

or, if y be the breadth of the orifice at the depth z,

$$Ku = \int_a^b y\sqrt{(u^2 + 2gz)}\, dz,$$

a and b being the depths of the upper and lower boundaries of the orifice.

If the motion be not steady, an approximate solution can be obtained when the orifice, although finite, is not large, by supposing the motion steady during any elementary interval of time, and taking, as in the previous case, the sum of the quantities of fluid passing through all the small orifices into which the whole aperture is divided.

EXAMPLES.

1. Find the time of emptying a frustum of a paraboloid of revolution with its axis vertical through a small orifice in the centre of the base.

2. Shew that the time in which a cone, the axis of which is inclined to the vertical, will be emptied through a hole at the vertex, is $At \div 5\kappa$, where A is the area of the surface of the fluid at first, κ of the orifice, and t is the time of falling freely through the entire vertical space described by the fluid.

3. Shew that the time of emptying a sphere is less than the time of emptying any spherical segment of equal capacity through a hole at the vertex.

4. The side of a vessel containing fluid is a plane inclined to the vertical, and small orifices are made along its line of intersection with a vertical plane at right angles to it; prove that all the parabolic jets are touched by two fixed straight lines.

5. A vessel of the form of a slender parallelopiped is filled with fluid, and placed upon a rough horizontal plane; determine at what height a given orifice must be made in one of its vertical sides, in order that the issuing jet may have the greatest tendency to overthrow the vessel.

6. In the vertical side of a vessel containing fluid an infinite number of small holes, bored perpendicular to the side, lie in a straight line inclined at an angle $\tan^{-1}\frac{1}{2}$ to the horizon: find the equation to the surface of the issuing fluid, and shew that any horizontal section of it is a circle.

7. A portion of a parabola, bounded by the curve, the axis, and the latus rectum, revolves about the latus rectum and generates a surface, which is placed with its axis of revolution vertical. If the vessel thus formed be filled with fluid, find the time in which it will empty itself through a small orifice at the lowest point.

8. A circular orifice is made in the horizontal base of a vessel containing fluid; if the fluid in the vessel is constantly

kept at the same height, the descending stream is bounded by the surface generated by the revolution of the curve $y^4x = \text{const.}$, about the axis of x.

9. Water is flowing steadily into a large reservoir through a straight tube of small section, inclined at a given angle to the vertical; having given the length of the tube, the depth of its lower extremity below the surface of the water in the reservoir, and the sections of both ends, find the rate at which water is flowing into the reservoir.

10. A vessel containing ink has a small hole pierced in one side, and is placed in a vessel of water; compare the velocity with which the ink will escape into the water, with that which it would have if it were flowing out into the air.

11. A closed cylindrical vessel one foot in height is half full of water, the other half being occupied by atmospheric air; if two small apertures be made, one at the base of the cylinder and the other five inches above it, shew that the density of the air in the vessel will decrease till it is $\left(1 - \frac{1}{12h}\right)$ times its original value approximately, and then increase again, h being the height of a water-barometer in feet.

12. A vertical cylinder is supplied with fluid at the top and loses it by an orifice at the bottom: assuming the motion to be by horizontal sections, and supposing the cylinder to be initially empty, g the accelerating force of gravity, v the constant velocity of the entering stream, and m the ratio of its transverse section to that of the cylinder, find the velocity of the issuing stream at any time t, and explain the result when $t = m\frac{v}{g}$.

13. A vessel in the form of a surface of revolution, the axis of which is vertical, has a small orifice at its vertex, and is filled with fluid; determine its form in order that the quantity of fluid which flows out in any time may vary as the square root of the time.

14. A vertical cylindrical vessel full of fluid has a fine crack extending along a generating line of the cylinder; find the time

of emptying a given portion of the cylinder. Examine the case in which the time of emptying the whole cylinder is required.

15. A right cone is filled with fluid and placed with a generating line horizontal, and uppermost, and a small orifice is made at the lowest point; find the time in which it will be emptied.

16. The surface of a vertical cylinder is pierced by a series of small holes in the form of a helix, the highest hole being at the top of the cylinder, and vertically above the lowest, and no other two holes being in the same vertical line. Determine the equation to the curve traced by the issuing fluid upon the horizontal plane passing through the lowest hole, the cylinder being kept constantly full.

Shew that the mean range is to the height of the cylinder as $\pi : 4$, and that the area included between the base of the cylinder and the curve above mentioned is

$$\frac{\pi h^2}{4}\left(\cot\alpha + \frac{8}{3}\right),$$

where α is the inclination of the line of holes to the horizon, and h the height of the cylinder.

17. A cubical vessel, having one side horizontal, is divided into two equal parts by a vertical partition, and one of the compartments is filled with fluid. If a small orifice be bored through the partition at a distance below the surface greater than half the depth of fluid, find the time which elapses before the fluid stands at the same height in both compartments.

18. A filament of fluid oscillates in a thin cycloidal tube of uniform bore, the axis of the cycloid being vertical and vertex downwards. Supposing the filament to be placed initially with its lower end at the lowest point of the tube, find the pressure at any point of the filament at any time.

Shew that the pressure is a maximum, during the whole motion, at the middle point of the filament.

19. A filament of fluid oscillates in a thin hypocycloidal tube of uniform bore under the action of a force tending to the centre of the fixed circle, and varying as the distance: supposing

the filament to be placed initially with one end at the vertex of the hypocycloid, find the pressure at any point of the filament at any time.

20. A small orifice of area κ is opened in the base of a vertical cylinder initially full of fluid. The fluid is forced through the orifice by a piston fitting the cylinder, to which is applied an uniform pressure P equal in amount to n times the weight of the fluid which the cylinder can contain. Shew that $\frac{1}{m}$th of the fluid will be evacuated in a time expressed by

$$2\left(\frac{A^2-\kappa^2}{\kappa^2}\cdot\frac{h}{2g}\right)^{\frac{1}{2}}\left\{\sqrt{(n+1)}-\sqrt{\left(n+1-\frac{1}{m}\right)}\right\},$$

where h is the height of the cylinder and A the area of its transverse section.

21. If the orifice of a conical vessel containing water be a section of the cone, perpendicular to its axis and at a distance δ from its vertex, and v be the velocity with which the water discharges itself, when its surface is at a distance z from the cone's vertex, prove that

$$\frac{dv^2}{dz}+\frac{(z+\delta)(z^2+\delta^2)}{z\delta^3}v^2=2g\frac{z^3}{\delta^3};$$

the axis of the vessel being vertical.

22. Two points are connected by a tube of small uniform bore through which heavy fluid is flowing steadily: the axis of the tube being in the vertical plane through the two points and its length being given, find its form when the whole pressure on the tube is a minimum.

CHAPTER X.

FURTHER APPLICATIONS OF THE EQUATIONS OF MOTION.

146. THE following proof of an important theorem is taken, with slight variations, from a paper, by Professor Stokes, in the eighth volume of the *Cambridge Philosophical Transactions.*

THEOREM. *Let the accelerating forces X, Y, Z, which act on the fluid, be such that $Xdx + Ydy + Zdz$ is the exact differential dV of some function of the co-ordinates. Then, if for the whole, or a certain portion of the fluid mass, the motion is at any one instant such that $udx + vdy + wdz$ is an exact differential, that expression will always be an exact differential, for the whole mass, or for the portion of fluid for which it was so at first.*

Suppose ρ a function of p, and let

$$\frac{1}{\rho} = f'(p);$$

Then the equations of fluid motion are,

$$\frac{df(p)}{dx} = X - \frac{du}{dt} - u\frac{du}{dx} - v\frac{du}{dy} - w\frac{du}{dz},$$

$$\frac{df(p)}{dy} = Y - \frac{dv}{dt} - u\frac{dv}{dx} - v\frac{dv}{dy} - w\frac{dv}{dz}.$$

$$\frac{df(p)}{dz} = Z - \frac{dw}{dt} - u\frac{dw}{dx} - v\frac{dw}{dy} - w\frac{dw}{dz}.$$

Differentiate the first of these equations with respect to y, and the second with respect to x, and subtract; then, putting

$$\frac{D}{Dt} \text{ for } \frac{d}{dt} + u\frac{d}{dx} + v\frac{d}{dy} + w\frac{d}{dz},$$

and observing that, since $Xdx + Ydy + Zdz$ is an exact differential, $\frac{dX}{dy} = \frac{dY}{dx}$,

we obtain

$$\frac{D}{Dt}\left(\frac{du}{dy}-\frac{dv}{dx}\right)+\frac{du}{dy}\,\frac{du}{dx}+\frac{dv}{dy}\,\frac{du}{dy}+\frac{dw}{dy}\,\frac{du}{dz}$$

$$-\frac{du}{dx}\,\frac{dv}{dx}-\frac{dv}{dx}\,\frac{dv}{dy}-\frac{dw}{dx}\,\frac{dv}{dz}=0.$$

Add and subtract $\frac{du}{dz}\,\frac{dv}{dz}$, and put

$$2\omega'=\frac{dw}{dy}-\frac{dv}{dz},\quad 2\omega''=\frac{du}{dz}-\frac{dw}{dx},\quad 2\omega'''=\frac{dv}{dx}-\frac{du}{dy};$$

then the above equation may be written

$$\frac{D\omega'''}{Dt}=\frac{du}{dz}\omega'+\frac{dv}{dz}\omega''-\left(\frac{du}{dx}+\frac{dv}{dy}\right)\omega'''.$$

Similarly,

$$\frac{D\omega''}{Dt}=\frac{dw}{dy}\omega''+\frac{du}{dy}\omega'-\left(\frac{dw}{dz}+\frac{du}{dx}\right)\omega'',$$

$$\frac{D\omega'}{dt}=\frac{dv}{dx}\omega''+\frac{dw}{dx}\omega'''-\left(\frac{dv}{dy}+\frac{dw}{dz}\right)\omega',$$

and it will be observed that, on account of the continuity of the motion, the differential coefficients $\frac{du}{dx}$ ...... cannot become infinite.

Suppose that when $t=0$, either there is no motion, or the motion is such that $udx+vdy+wdz$ is a perfect differential. This may be the case for the whole or for any portion of the fluid mass.

Then initially,

$$\omega'=0,\quad \omega''=0,\quad \omega'''=0.$$

Now let L be a superior limit to the numerical values of the coefficients of $\omega'\ \omega''\ \omega'''$; then, whether $\omega'\ \omega''\ \omega'''$ are, any or all

of them, subsequently positive or negative, they cannot be numerically greater than if they satisfy the equations,

$$\frac{D\omega'}{Dt} = \pm L\,(\omega' + \omega'' + \omega'''),$$

$$\frac{D\omega''}{Dt} = \pm L\,(\omega' + \omega'' + \omega'''),$$

$$\frac{D\omega'''}{Dt} = \pm L\,(\omega' + \omega'' + \omega'''),$$

the signs being taken $+$ or $-$, according as it is conceived possible that, at any subsequent time, any one of the quantities may be positive or negative.

In any case, add the equations together,

then $$\frac{d\Omega}{dt} = rL\Omega,$$

where $\Omega = \omega' + \omega'' + \omega'''$, and $r = \pm 1$, or ± 3;

$$\therefore\ \Omega = C\epsilon^{rLt};$$

$$\text{initially, } \Omega = 0, \quad \therefore\ C = 0,$$

$$\text{and } \omega' + \omega'' + \omega''' = 0, \text{ always} \ldots\ldots\ldots\ldots\ldots \quad (3).$$

But we obtain, by adding or subtracting the equations

$$\pm\frac{D\omega'}{Dt} = \pm\frac{D\omega''}{Dt} = \pm\frac{D\omega'''}{Dt},$$

$$\pm\,\omega' = \pm\,\omega'' = \pm\,\omega''', \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4),$$

since these quantities are initially zero;

and, in whatever order the signs are taken, the equations (3) and (4) give

$$\omega' = 0, \quad \omega'' = 0, \quad \omega''' = 0.$$

Hence, a fortiori, the actual values of ω', ω'', ω''', must be equal to zero; and therefore $udx + vdy + wdz$ is always, if once, a perfect differential.

147. In the paper referred to on the Friction of Fluids in Motion, and also in the *Cambridge and Dublin Mathematical Journal*, Vol. III., Professor Stokes has fully discussed the various proofs which have been given of this theorem.

The proof first given, by Lagrange, depends on the possibility of expanding u, v, and w in positive integral powers of t, for small values of t. Mr Power (*Cambridge Philosophical Transactions*, Vol. VII. Part III.) has extended the proof to cases in which u, v, and w are capable of expansion according to any positive powers of t.

This proof, however, is incomplete, as it does not include functions, (such as $t \log t$, $\epsilon^{-\frac{1}{t^2}}$), which cannot be expanded in positive powers of t.

It appears also that Lagrange's proof would apply to the case in which the variation of pressure in different directions is taken account of, in which case the theorem is not true, and that the same objection applies to Poisson's proof.

A proof has been given by Cauchy, apparently free from objection, although of considerable length, in the *Mémoire sur la Theorie des Ondes*, which appeared in the first Volume of the *Mémoires Presentés a l'Institut*. This proof is given in full, in the *Cambridge and Dublin Mathematical Journal*, Vol. III. p. 210.

Physical Interpretation.

148. The cases in which $udx + vdy + wdz$ is or is not a perfect differential, represent states of motion of distinctly different characters, as the following article, also taken from the paper of Professor Stokes, will shew.

Conceive an indefinitely small element of a fluid in motion to become suddenly solidified, and the fluid about it to be suddenly destroyed; let the form of the element be so taken that the resulting solid shall be that which is the simplest with respect to rotatory motion, namely, that which has its three principal moments about axes passing through the centre of gravity equal to each other, and therefore every axis passing through that

point a principal axis, and consider the linear and angular motions of the element immediately after solidification.

By the instantaneous solidification velocities will be suddenly generated or destroyed in the different portions of the element, and a set of impulsive forces will be called into action. Let x, y, z be the co-ordinates of the centre of gravity G of the element at the instant of solidification, $x+x'$, $y+y'$, $z+z'$ those of any other point in it.

Let u, v, w be the velocities of G along the three axes just before solidification, u', v', w' the relative velocities of the point whose relative co-ordinates are x', y', z'.

Let $\bar{u}$, $\bar{v}$, $\bar{w}$ be the velocities of G, $u_{,}$, $v_{,}$, $w_{,}$ the relative velocities of the point $(x'y'z')$, and ω', ω'', ω''' the angular velocities just after solidification.

Since all the impulsive forces are internal,

$$\bar{u} = u, \quad \bar{v} = v, \quad \bar{w} = w.$$

Also, by the conservation of angular momentum,

$$\Sigma m \{y'(w_{,} - w') - z'(v_{,} - v_{,}')\} = 0, \text{ \&c.}$$

m denoting an element of the mass considered.

But
$$u_{,} = \omega'' z' - \omega''' y',$$

$$u' = \frac{du}{dx} x' + \frac{du}{dy} y' + \frac{du}{dz} z', \text{ ultimately,}$$

and similar expressions hold good for the other quantities.

Substituting in the above equations, and observing that

$$\Sigma (my'z') = 0, \quad \Sigma (mz'x') = 0, \quad \Sigma (mx'y') = 0, \text{ and}$$

$$\Sigma (mx'^2) = \Sigma (my'^2) = \Sigma (mz'^2), \text{ we have}$$

$$2\omega' = \frac{dw}{dy} - \frac{dv}{dz}, \quad 2\omega'' = \frac{du}{dz} - \frac{dw}{dx}, \quad 2\omega''' = \frac{dv}{dx} - \frac{du}{dy}.$$

We see then that an indefinitely small element of the fluid, of which the three principal moments about the centre of gravity are equal, if suddenly solidified and detached from the rest of the fluid, will begin to move with a motion simply of translation, which may however vanish, or a motion of translation combined

with one of rotation, according as $udx + vdy + wdz$ is or is not an exact differential.

149. *Lines of motion.* The direction of the motion of the fluid particle at the point x, y, z is defined by the quantities u, v, w, expressing the velocities at that point, and therefore the differential equations of the lines of motion are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w};$$

and it is obvious that these lines intersect at right angles the surfaces of which the differential equation is

$$udx + vdy + wdz = 0.$$

Consider, as a particular case, *the steady motion of an incompressible fluid in two dimensions, when $udx + vdy$ is a perfect differential.*

From the given condition, we have

$$\frac{du}{dy} = \frac{dv}{dx} \quad \dots\dots\dots\dots\dots\dots\dots\dots \quad (\alpha),$$

and, from the equation of continuity,

$$\frac{du}{dx} + \frac{dv}{dy} = 0 \quad \dots\dots\dots\dots\dots\dots\dots\dots \quad (\beta).$$

The differential equation of the lines of motion is

$$vdx - udy = 0,$$

which, by the equation (β), is a perfect differential.

Let $P = C$ be the integral, that is,

let $$v = \frac{dP}{dx}, \text{ and } -u = \frac{dP}{dy},$$

$$\therefore \text{ from } (\alpha), \ \frac{d^2P}{dx^2} + \frac{d^2P}{dy^2} = 0 \quad \dots\dots\dots\dots\dots\dots \quad (\gamma),$$

an equation which P must satisfy.

If a possible value of P be found, and the values of u and v be obtained, the pressure is given by the equation,

$$\frac{p}{\rho} = S - \frac{1}{2}(u^2 + v^2) + C,$$

where C is constant along any particular line of motion.

Suppose, if possible, that the lines of motion are similar hyperbolas, given by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = m^2,$$

then we must have, from (γ),

$$\frac{1}{a^2} - \frac{1}{b^2} = 0,$$

and the hyperbolas must be equilateral.

Taking then $P = \mu (x^2 - y^2)$, we have

$$u = 2\mu y, \quad v = 2\mu x,$$

and therefore the velocity $\sqrt{(u^2 + v^2)}$ varies as the distance from the origin.

The general integral of the equation (γ), is

$$P = \phi \{x + y\sqrt{(-1)}\} + \psi \{x - y\sqrt{(-1)}\},$$

and it is obvious, by taking

$$\phi(\alpha) = \psi(\alpha) = \frac{1}{2}\mu\alpha^2,$$

that $\mu (x^2 - y^2)$ is a particular form of the integral.

150. If the motion of the fluid particles in one plane be symmetrical with regard to a centre, the equation of continuity can be at once integrated, when the fluid is inelastic.

The lines of motion are radii from the centre, and the velocity (V) is a function of r; we have then

$$u = V\frac{x}{r}, \quad v = V\frac{y}{r},$$

$$\frac{du}{dx}=\frac{x}{r}\frac{dV}{dr}\frac{dr}{dx}+\frac{V}{r}-\frac{Vx}{r^2}\frac{dr}{dx}$$

$$=\frac{dV}{dr}\frac{x^2}{r^2}+\frac{V}{r}-\frac{Vx^2}{r^3},$$

$$\frac{du}{dy}=\frac{dV}{dr}\frac{y^2}{r^2}+\frac{V}{r}-\frac{Vy^2}{r^3},$$

and the equation of continuity is

$$\frac{dV}{dr}+\frac{V}{r}=0, \quad \text{or } \frac{d}{dr}(rV)=0.$$

Hence we obtain

$$rV=f(t),$$

and therefore, at any given time, the velocity is inversely proportional to the distance.

If the motion be steady, $f(t)$ is constant.

151. PROP. *The fluid being incompressible, and its motion symmetrical, in all directions, with regard to a centre, it is required to find the integral of the equation of continuity.*

The velocity is a function of the time and of the distance from the centre: hence taking V as the velocity at (x, y, z),

$$u=V\frac{x}{r}, \quad v=V\frac{y}{r}, \quad w=V\frac{z}{r},$$

where $$r^2=x^2+y^2+z^2;$$

and the equation of continuity is

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0.$$

Now $$\frac{du}{dx}=\frac{dV}{dr}\frac{x^2}{r^2}+\frac{V}{r}-V\frac{x^2}{r^3},$$

$$\frac{du}{dy}=\frac{dV}{dr}\frac{y^2}{r^2}+\frac{V}{r}-V\frac{y^2}{r^3},$$

$$\frac{dw}{dz}=\frac{dV}{dr}\frac{z^2}{r^2}+\frac{V}{r}-V\frac{z^2}{r^3};$$

therefore the equation of continuity becomes

$$\frac{dV}{dr} + \frac{2V}{dr} = 0,$$

or $$\frac{d}{dr}(r^2 V) = 0,$$

and therefore $r^2 V = f(t)$.

This equation expresses that the quantity of fluid which during a small interval of time, at a given epoch, flows across a spherical surface, is the same whatever be the radius; an obvious condition of the continuity of the motion.

152. If the fluid be compressible, or heterogeneous, and the motion symmetrical about a centre, the equation of continuity, since

$$\frac{d\rho}{dx} = \frac{d\rho}{dr}\frac{x}{r}, \quad \frac{d\rho}{dy} = \frac{d\rho}{dr}\frac{y}{r}, \quad \text{and} \quad \frac{d\rho}{dz} = \frac{d\rho}{dr}\frac{z}{r},$$

becomes

$$\frac{d\rho}{dt} + V\frac{d\rho}{dr} + \frac{\rho}{r^2}\frac{d}{dr}(r^2 V) = 0.$$

This may be obtained from the form in Art. (123), by observing that k is explicitly a function of s *only*, that is, of r, and varies as r^2.

153. *A vessel containing liquid moves vertically upwards with an uniform acceleration; required to find the pressure at any point.*

The vessel may be supposed to be raised by a string passing over a pulley, and having at its other end a weight greater than the weight of the vessel of fluid.

Let f be the upward acceleration, and therefore mf the effective force on a particle m of fluid.

Measuring z downwards, and reversing the effective forces,

$$dp = \rho\,(g + f)\,dz,$$

and $$p = C + \rho\,(g + f)\,z.$$

Let the pressure at the free surface be supposed constant and be represented by Π; then, if z', and $z' + x$, be the vertical

distances from the origin of the free surface and of any other horizontal plane in the fluid,

$$\Pi = C + \rho (g+f) z',$$
$$p = C + \rho (g+f)(z'+x),$$

and therefore $$p = \rho (g+f) x + \Pi.$$

This result might also have been obtained by arguing that the resultant fluid pressure on any portion, elementary or finite, of the fluid, producés, in combination with the force of gravity, an upward acceleration f, and therefore that forces mf, acting vertically downwards, would produce the same pressure at any point of the fluid, supposed at rest. By such reasoning the problem is at once, apparently without the intervention of D'Alembert's principle, placed in the domain of Hydrostatics, and, taking axes fixed relative to the fluid, the equilibrium equation becomes applicable, and leads to the value of p just obtained.

The reasoning of Art. (28), in which the equilibrium of a revolving fluid is discussed, is of the same kind, and it must be noticed that in each case an assumption is made, which is really equivalent to the application of D'Alembert's principle, although for these cases the enunciation of the principle in its most general form is unnecessary. It is in fact assumed implicitly, that, when there is no relative displacement of the fluid particles, the molecular actions are the same as if the fluid were at rest in the same form, or that, if it be conceived possible that the motion would call into play additional molecular actions, no alteration is produced in the pressure by such actions, and the pressure consequently depends on the force of gravity and the hypothetical forces mf in the present case, and, in the case of Art. (28), on the hypothetical forces $m\omega^2 r$ in combination with the force of gravity.

154. *A vessel containing incompressible fluid moves vertically upwards with a given acceleration, and the fluid rotates uniformly about a vertical axis; it is required to find the pressure at any point.*

Let f be the acceleration and ω the angular velocity; then, by the preceding article, and by Art. (28), we may suppose the

fluid at rest and maintained in its state of relative equilibrium by the action of gravity, of the forces mf downwards, and of the forces $m\omega^2 r$ perpendicular to the axis of rotation.

Taking this axis as the axis of z, we have

$$dp = \rho \{\omega^2 (xdx + ydy) - (g + f)\, dz\},$$

and therefore $$p + C = \rho \frac{\omega^2}{2} (x^2 + y^2) - \rho (g + f)\, z,$$

the constant being determined by the particular circumstances of the case.

155. PROP. *To find the rate of efflux through a small orifice in the base of a vessel in motion, as in Art.* (153).

If the orifice be very small compared with the upper surface of the fluid, we may suppose during any small time that the motion is relatively steady, and therefore that the motion of the fluid would be the same as in a vessel at rest, if the quantity g be replaced by $g + f$.

We thus obtain, if h be depth of the orifice, and v the velocity of efflux, relative to the vessel,

$$v^2 = 2\, (g + f) h.$$

Or we may reason as follows:

The equation of motion, measuring z downwards, is

$$\frac{dp}{\rho} = gdz - \frac{d\,(u)}{dt}\, ds,$$

if we take u as the actual velocity at any point of the fluid.

Let v be the velocity, relative to the vessel, at the point; then v is a function of s only, and

$$u = v - ft \frac{dz}{ds}$$

if the vessel be supposed to have started from rest;

$$\therefore \frac{d\,(u)}{dt} = v \frac{dv}{ds} - f \frac{dz}{ds},$$

$$\frac{dp}{\rho} = (g + f)\, dz - v \frac{dv}{ds} . ds,$$

and $$\frac{p}{\rho} + C = (g+f)\,z - \tfrac{1}{2}v^2.$$

Let z' be the depth of the upper surface;

therefore $$\frac{1}{\rho}\Pi + C = (g+f)\,z' - \tfrac{1}{2}V^2,$$

$$\frac{1}{\rho}\Pi + C = (g+f)(z'+h) - \tfrac{1}{2}v^2,$$

$$v^2 = V^2 + 2(g+f)\,h,$$

v and V being the velocities at the orifice and at the upper surface, and, neglecting V^2, we get the same result as before.

If the vessel be made to ascend with the acceleration f, the quantity of fluid which issues through the orifice in a given time can be approximately determined as in Art. 129. If, however, the vessel be raised by a string passing over pullies, and supporting a given weight, we must observe that the acceleration decreases as the quantity of fluid in the vessel is diminished.

In the latter case, let W be the given weight, and W' the weight of the vessel and fluid together; also let U be the quantity of fluid lost in the time t, and X the area of the surface at the end of the same time.

We have therefore, measuring x, the height of the surface, from the orifice,

$$dU = -Xdx,$$

and the weight of the fluid and vessel together at the time t,

$$= W' - \int \rho\, dU = W' + \rho \int Xdx;$$

but $dU = k v\, dt$, if x be the area of the orifice,

and the acceleration $= g \cdot \dfrac{W - W' - \rho\int Xdx}{W + W' + \rho\int Xdx}$;

$$\therefore\ v^2 = 2gx\,\frac{2W}{W + W' + \rho\int Xdx}.$$

Hence, $$-Xdx = 2k\sqrt{(gx)}\left(\frac{W}{W + W' + \rho\int Xdx}\right)^{\frac{1}{2}} . dt,$$

from which the relation between x and t can be found if X be known in terms of x.

156. *In the side of a vessel containing fluid which rotates uniformly about a vertical axis, a small aperture is made; required to find the velocity of efflux.*

Assuming that the motion is steady, let u be the velocity at any point relative to the fluid, that is, that part of the velocity which does not depend upon the rotation; then the equation of steady motion is

$$\frac{p}{\rho} = C - \frac{1}{2}\rho\,(\omega^2 r^2 + u^2) - gz,$$

neglecting the motion at the upper surface, we have

$$\frac{\Pi}{\rho} = C - \frac{1}{2}\rho\omega^2 r^2 - gz,$$

as the equation to that surface, and, if r' z' be co-ordinates of a point in the orifice, we have, at the orifice,

$$\frac{\Pi}{\rho} = C - \frac{1}{2}\rho\,(\omega^2 r'^2 + v^2) - gz';$$

$$\text{and } \frac{\Pi}{\rho} = C - \frac{1}{2}\rho\omega^2 r'^2 - gz, \text{ at the surface,}$$

$$\text{and therefore, } v^2 = 2g\,(z' - z),$$

an equation which determines the velocity at the orifice.

157. *A closed vessel, the interior surface of which is spherical, is filled with heavy inelastic fluid, and the vessel is moved in any way; it is required to find, at any instant, the surfaces of equal pressure.*

Supposing the surface smooth and the fluid initially at rest, it is clear that no rotation can be caused in the fluid, and therefore that the actual motion of every particle of the fluid will be the same as that of the centre of the sphere. At any given instant, let f be the acceleration, in a known direction, of the centre of the sphere; then, by D'Alembert's principle, the fluid may be supposed at rest under the action of gravity and the

reversed forces mf, and, since the resultant of the acceleration g, and the reversed acceleration f, is the same, both in magnitude and direction, for all particles of the fluid, it follows that the surfaces of equal pressure are, at the given instant, planes perpendicular to the direction of that resultant.

158. *An infinite mass of homogeneous incompressible fluid acted upon by no forces is at rest, and a spherical portion of the fluid is suddenly annihilated; it is required to find the instantaneous alteration of pressure at any point of the mass, and the time in which the cavity will be filled up, the pressure at an infinite distance being supposed to remain constant.*

Assuming that the motion of the fluid is symmetrical with regard to the centre of the spherical space, the equation of continuity gives the relation,

$$r^2 V = F(t),$$

between the distance from the centre, and the velocity at that distance.

Taking $\qquad V = \dfrac{d\phi}{dr}$, we obtain

$$\phi = \psi(t) - \frac{1}{r} F(t);$$

but, initially, $V = 0$ for all values of r;

$\therefore$ $F(0) = 0$, and the initial value of ϕ is $\psi(0)$.

From the equation of motion,

$$\frac{1}{\rho} dp = -\frac{1}{2} d \, . \, V^2 - d \, . \, \frac{d\phi}{dt},$$

we have

$$\frac{p}{\rho} = \chi(t) - \frac{1}{2} V^2 - \frac{d\phi}{dt}$$

$$= f(t) - \frac{1}{2} V^2 + \frac{F'(t)}{r}.$$

Let ϖ be the pressure at an infinite distance, then since when $r = \infty$, $V = 0$,

$$\frac{\varpi}{\rho} = f(t),$$

and $$\frac{p}{\rho} = \frac{\varpi}{\rho} - \frac{1}{2} V^2 + \frac{F'(t)}{r}.$$

If a be the radius of the spherical portion of fluid annihilated; then, initially, when $r = a$, $V = 0$, and $p = 0$;

$$\therefore\ 0 = \frac{\varpi}{\rho} + \frac{F'(0)}{a},$$

and, at a distance r, the initial value of p is

$$\frac{\varpi}{\rho}\left(1 - \frac{a}{r}\right).$$

At the time t, let r be the radius of the free surface, which is assumed to be spherical;

$$\text{hence,}\quad 0 = \frac{\varpi}{\rho} - \frac{1}{2}V^2 + \frac{F'(t)}{r}; \ldots\ldots\ldots\ldots\ldots (\alpha),$$

but, from the equation $F(t) = r^2 V$,

$$F'(t) = r^2 \frac{dV}{dt} + 2rV\frac{dr}{dt} = r^2 V\frac{dV}{dr} + 2rV^2,$$

$$\text{observing that } V = \frac{dr}{dt}.$$

Substituting in the equation (α),

$$rV\frac{dV}{dr} + \frac{3}{2}V^2 + \frac{\varpi}{\rho} = 0,$$

and, integrating,

$$r^3 V^2 = \frac{2}{3}\,\frac{\varpi}{\rho}\,(a^3 - r^3),$$

$$\text{or}\qquad \frac{dt}{dr} = -\sqrt{\left(\frac{3\rho}{2\varpi}\right)}\,\frac{r^{\frac{3}{2}}}{\sqrt{(a^3 - r^3)}}.$$

Hence, the time in which the cavity will be filled up is given by the equation,

$$t = \sqrt{\left(\frac{3\rho}{2\varpi}\right)}\int_0^a \frac{r^{\frac{3}{2}}dr}{\sqrt{(a^3 - r^3)}},$$

which, by putting $r = az^2$, may be written

$$t = \sqrt{\left(\frac{6\rho}{\varpi}\right)}\,.\,a\int_0^1 \frac{z^4 dz}{\sqrt{(1 - z^6)}}.$$

Senate-House Problems, 1847.

EXAMPLES.

1. A mass of fluid moves in such a way that each particle describes a circle in one plane about a fixed axis, shew that the equation of continuity is

$$\frac{d\rho}{dt} + \frac{d(\rho\omega)}{d\theta} = 0,$$

where ω is the angular velocity of a particle whose vectorial angle, measured from a line in the plane of its motion through the fixed axis, is θ at the time t.

2. A mass of fluid is in motion so that each particle moves in a cylinder about a fixed axis; find the equation of continuity.

3. Steam is rushing from a boiler through a conical pipe, the diameters of the extremities of which are D and d respectively; if V and v be the corresponding velocities of the steam, shew that

$$v = V \cdot \frac{D^2}{d^2} \cdot \epsilon^{\frac{v^2 - V^2}{2k}},$$

where k is the pressure divided by the density, and supposed constant. The motion may be supposed to be that of fluid diverging from a centre, the centre being the vertex of the cone of which the pipe forms a portion.

4. Each particle of a mass of incompressible fluid moves in a plane through the axis of z; find the equation of continuity.

5. If r, θ, be the polar co-ordinates of a point at which the density is ρ, and u, v, the velocities along, and perpendicular to the radius vector, shew that the equation of continuity for motion in one plane, is

$$\frac{d(\rho r u)}{dr} + \frac{d(\rho u)}{d\theta} + r\frac{d\rho}{dt} = 0.$$

6. Two vessels, containing homogeneous incompressible fluids, acted on by no forces, but subject to a given external pressure, are connected by a cylindrical tube of small bore. A portion of the fluid in the tube being supposed to be suddenly

annihilated, determine the instantaneous change of pressure, and the subsequent motion in the tube.

7. In the case of the steady motion of an incompressible fluid in one plane, shew that a system of parabolas, having a common focus and coincident axes, is a possible system of lines of motion. Shew that the velocity at any point varies inversely as the square root of the focal distance.

8. An uniform semicircular tube stands in a vertical plane with its open ends resting in a vessel of fluid. One-third of its length is occupied with air, and the remainder with the fluid. Find the time of a small vibration caused by an instantaneous increase of the pressure of the fluid, considering the density of the air in the tube at any time to be uniform.

9. A closed vessel, filled with elastic fluid, is moved, in a vertical direction, with a given acceleration; find what the law of the density must be in order that the fluid particles may be, relatively to each other, at rest.

10. The bob of a pendulum is a hollow sphere which is filled with fluid; find the surfaces of equal pressure for any position of the pendulum.

11. A closed vessel is filled with water containing in it a piece of cork which is free to move; if the vessel be suddenly moved forwards by a blow, shew that the cork will shoot forwards relatively to the water.

12. A closed vessel is filled with water which is at rest, and the vessel is then moved in any manner; apply the principle of the conservation of areas to prove that, if the vessel have any motion of rotation, no finite portion of the water can remain at rest relatively to the vessel.

13. A swollen river flows through two bridges, first through A, then through B, the breadth of its channel being considerably narrower at each bridge than between them, where the section of the trough of the river is a constant rectangle; the river is at its greatest height so that the motion may be considered steady:

consider (i) the effect of the variation of the normal section on the velocity of the main current, (ii) the corresponding variation of the pressure.

(iii) Shew that the surface of the fluid in motion will not be horizontal, (iv) that there will be parts of the fluid C, C near the banks between A and B which have no velocity parallel to AB, and that there will be side currents from C, C' towards A, as well as towards B.

14. Two rigid laminæ, in one of which is a small circular aperture, are placed very near to each other with their planes parallel. Supposing air to be rushing through the aperture, shew that the differential equation of its motion is

$$\frac{d^2v}{dt^2} = \kappa\left(\frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} - \frac{v}{r^2}\right) - \frac{d}{dr}\left(\frac{d \, . \, v^2}{dt} + v^2\frac{dv}{dr}\right).$$

15. A mass of homogeneous incompressible fluid, moving in a straight tube of uniform bore under the action of no forces, meets a piston which, by compressing a spring, gradually reduces the fluid to rest: if p be the pressure (on a unit of area) at any point of the fluid, whose distance, from the extremity (E) not in contact with the piston, is x; shew that, at any time t,

$$p = \frac{P}{V}x,$$

V being the volume of the fluid, and P the pressure exercised by the piston, at the time t, upon the extremity (E') of the fluid in contact with it.

It has been asserted that, when an incompressible fluid mass, moving in a tube, is suddenly reduced to rest by an impulsive force, the pressure caused by the impulse is the same at every point of the fluid: prove that this is not true.

16. In the preceding problem, if the bore of the tube be variable and small, shew that

$$p = \frac{P\int_0^x X dx}{\kappa \int_0^u X dx},$$

where κ is the area of the piston, $\frac{1}{X}$ the area of the section of the tube at the distance x from (E), and u the distance of E' from E.

17. A homogeneous fluid is contained between two concentric spherical surfaces, the radius of the inner being a and that of the outer indefinitely great. The fluid is attracted to the centre of these surfaces by a force $\phi(r)$, and a constant pressure Π is exerted at the outer surface.

Suppose $\int \phi(r)\, dr = \psi(r)$, and that $\psi(r)$ vanishes when r is infinite. Shew that if the inner surface is suddenly removed, the pressure at the distance r is suddenly diminished by

$$\Pi \frac{a}{r} - \frac{a\rho}{r} \psi(a).$$

Find $\phi(r)$ so that the pressure immediately after the inner surface is removed may be the same as it would be if no attractive force existed. Also with this value of $\phi(r)$, find the velocity of the inner boundary of the fluid at any period of the motion.

CHAPTER XI.

RESISTANCES.

159. WHEN a solid body is dragged through water at rest, or is held at rest immersed in a stream, a certain force must be exerted in order to produce the motion in the former case, or to maintain the equilibrium in the latter. This force measures the resistance of the fluid.

These two cases may be looked upon as the same, for if we impress on the body at rest and the stream a velocity equal and opposite to that of the stream, the second case is reduced to the first; and in general the resistance of a fluid on a solid within it will depend on the relative velocity of the fluid and the solid.

The ordinary theory of resistances is given in the following article.

160. PROP. *A plane lamina is immersed in a stream, perpendicular to the direction of its motion; it is required to find the pressure on the lamina.*

Suppose the motion of the fluid steady, and let $mf'(s)$ be the force acting on a particle m of fluid, s being measured along a line of motion. We have then from Art. 127, the relation

$$\frac{p}{\rho} = f(s) - \frac{1}{2}v^2 + C,$$

between the pressure and the velocity at any point.

Now at a certain distance from the plane, and beyond it, we may suppose that the pressure and velocity are not affected by the presence of the plane in the fluid. If u' be the velocity and p' the pressure at some point beyond this distance, at which $s = a$, we have

$$\frac{p'}{\rho} = f(a) - \frac{1}{2}u'^2 + C.$$

At the plane let $s = 0$, and assume that the velocities of the

particles of fluid in immediate contact with the plane are destroyed; then, if ϖ' be the pressure at a point of the plane,

$$\frac{\varpi'}{\rho} = f(0) + C,$$

$$\text{and } \therefore \frac{\varpi' - p'}{\rho} = f(0) - f(a) + \frac{1}{2} u'^2.$$

If we consider the motion of the fluid before the immersion of the lamina, and take ϖ and u for the pressure and velocity at the same point, we have

$$\frac{\varpi}{\rho} = f(0) - \frac{1}{2} u^2 + C,$$

$$\text{and } \frac{\varpi - p'}{\rho} = f(0) - f(a) - \frac{1}{2} u^2 + \frac{1}{2} u'^2.$$

Hence $\varpi' - \varpi = \frac{1}{2} \rho u^2$, and this is the difference of the fluid pressures before and after the immersion of the lamina. The usual theory of resistances assumes that the pressure at the opposite side of the lamina is the same as if the lamina were not immersed, and therefore, if the velocities at all points of the position occupied by the lamina be the same before its immersion, the *resistance* upon it is $\frac{1}{2} \rho u^2$ (area of lamina).

161. *A stream flows obliquely against a plane; it is required to find the impelling force on the plane.*

Taking the velocities at all points of the stream the same, it may be assumed that no change will be produced in the pressure by moving the plane perpendicularly to the direction of the stream. The resistance therefore depends upon the velocity of the stream resolved in the direction perpendicular to the plane.

Hence if u be the velocity, and θ the angle between the direction of the stream and the normal to the plane, the resultant pressure on the lamina, normal to its plane,

$$= \frac{1}{2} \kappa \rho u^2 \cos^2 \theta,$$

where κ is the area of the lamina.

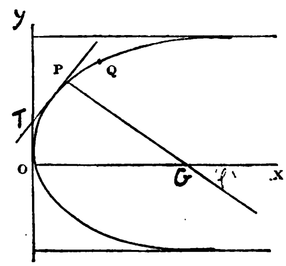
The pressures on the lamina, parallel and perpendicular to the stream, are respectively

$$\frac{1}{2}\kappa\rho u^2\cos^3\theta,\quad\text{and}\quad\frac{1}{2}\kappa\rho u^2\sin\theta\cos^2\theta.$$

If in either of the preceding cases the lamina have any motion in the direction of, or opposite to, the motion of the stream, it must be remembered that the resistance is normal to the plane, and depends on the relative normal velocity of the stream and the plane. If v be this relative velocity the resistance

$$=\frac{1}{2}\rho v^2\kappa.$$

162. *A cylindrical surface, bounded by planes parallel and perpendicular to its generating line, is immersed in a stream flowing in a direction perpendicular to the generating lines; it is required to find the impelling force of the stream on the surface.*



Let OPQ be a section by a plane perpendicular to the generating lines, O being the point at which the tangent is perpendicular to the direction, Ox, of the stream.

Take ϕ as the angle which the normal PG makes with Ox, or the tangent PT with Oy, and let $PQ = \delta s$, and $h =$ the height of the cylinder.

The force on PQ in direction $PG = \frac{1}{2}\rho hu^2\cos^2\phi\delta s$, and therefore the resultant force in the direction Ox

$$=\frac{1}{2}\rho hu^2\int\cos^3\phi ds,$$

and in the direction Oy the force

$$= \frac{1}{2} \rho h u^2 \int \cos^2 \phi \sin \phi ds.$$

If (x, y) be the co-ordinates of P, $\cos \phi = \frac{dy}{ds}$, $\sin \phi = \frac{dx}{ds}$, and the forces in directions Ox, Oy, are respectively

$$\frac{1}{2} \rho h u^2 \int \left(\frac{dy}{ds}\right)^2 dy, \text{ and } \frac{1}{2} \rho h u^2 \int \left(\frac{dy}{ds}\right)^2 \frac{dx}{dy} dy.$$

The limits of integration are obtained by drawing tangents parallel to Ox, or, if the curvature be not continued, by drawing lines parallel to Ox through the points of the curve farthest from Ox.

If the intrinsic equation to the curve, $s = f(\phi)$, be given, the expressions take the forms

$$\frac{1}{2} \rho h u^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3 \phi f'(\phi)\, d\phi, \quad \frac{1}{2} \rho h u^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \phi \sin \phi f'(\phi)\, d\phi.$$

Ex. 1. Suppose the surface that of a circular cylinder.

Then $y^2 = 2ax - x^2$ and $\frac{dy}{ds} = \frac{a-x}{a}$, if a be the radius;

$$\therefore \int_{-a}^{a} \left(\frac{dy}{ds}\right)^2 dy = \int_{-a}^{a} \frac{a^2 - y^2}{a^2} dy = \frac{4}{3} a,$$

and the force of the stream $= \frac{2}{3} \rho h a u^2$.

Ex. 2. Suppose the curve a parabola bounded by a double ordinate, the axis of the curve being in the direction of the stream.

The intrinsic equation is $\frac{ds}{d\phi} = \frac{2a}{\cos^3 \phi}$,

and the force parallel to $x = \frac{1}{2} \rho h u^2 \int_{-\beta}^{\beta} 2a d\phi = 2\rho h a u^2 \beta$,

β being the angle between the tangent at the extremity of the ordinate and the tangent at the vertex.

163. *A solid is generated by the revolution of a curve round an axis Ox; it is required to find the impelling force on the solid of a stream flowing in the direction Ox.*

In the figure of the preceding article, let OPQ be the generating curve; then $2\pi y\delta s$ is the surface generated by PQ, and the resultant force upon this element is evidently in the direction Ox, and

$$= \frac{1}{2}\rho u^2 \int \cos^3\phi 2\pi y ds$$

$$= \pi\rho u^2 \int y \cos^3\phi ds = \pi\rho u^2 \int_0^c y \left(\frac{dy}{ds}\right)^3 dy,$$

c being the extreme value of y.

If, for instance, the solid be a sphere, this expression

$$= \pi\rho u^2 \int_0^a y \frac{a^2 - y^2}{a^2} dy = \frac{1}{4}\pi\rho a^2 u^2.$$

164. *Resistance on a surface of any form.*

Take the axis of x parallel to the direction of the stream, and let l, m, n, be the direction cosines of the normal at any point (x, y, z) of the surface, and δS an element of the surface about the point.

The normal resistance on $\delta S = \frac{1}{2}\rho l^2 u^2 \delta S$, and therefore the whole resistance parallel to $x = \frac{1}{2}\rho u^2 \iint l^3 dS$.

Now $l\delta S = \delta y \delta z$, and $l = \frac{df}{dx} \div \sqrt{\left\{\left(\frac{df}{dx}\right)^2 + \left(\frac{df}{dy}\right)^2 + \left(\frac{df}{dz}\right)^2\right\}}$,

if $f(x, y, z) = 0$ be the equation to the surface;

$$\therefore \text{ the resistance parallel to } x = \frac{1}{2}\rho u^2 \iint \frac{\left(\frac{df}{dx}\right)^3 dy\,dz}{\left(\frac{df}{dx}\right)^2 + \left(\frac{df}{dy}\right)^2 + \left(\frac{df}{dz}\right)^2},$$

and similar expressions can be found for the resistances in the directions of y and z.

The limits of integration are defined by the trace on the plane yz of the cylinder, having its axis parallel to the axis of x, which envelops the surface.

165. *A heavy sphere descends vertically in a fluid; it is required to determine its motion.*

If a be the radius of the sphere, ρ and σ the densities of the fluid and the sphere, and v its velocity after having descended through a space x, the equation of motion is

$$\frac{4}{3}\pi\sigma a^3 v\frac{dv}{dx}=\frac{4}{3}\pi a^3 g(\sigma-\rho)-\frac{1}{4}\pi\rho a^2 v^2;$$

$$\text{or, } \frac{d.v^2}{dx}+\frac{3}{8}\frac{\rho}{\sigma}\frac{v^2}{a}=2g\frac{\sigma-\rho}{\sigma},$$

a linear equation, from which we obtain

$$v^2=\frac{16}{3}\frac{\sigma-\rho}{\rho}ga\,(1-\epsilon^{-\frac{3}{8}\frac{\rho}{\sigma}\frac{x}{a}}),$$

if the sphere have no initial velocity.

As x increases, v approximates to a terminal value,

$$\sqrt{\left(\frac{16}{3}\frac{\sigma-\rho}{\rho}ga\right)}.$$

It thus appears that, if a be very small, v is very small, and moreover that after a short descent the difference between the actual velocity and the terminal velocity becomes insensible.

Suppose that the sphere is of lead, and that its radius is $\frac{1}{100}$th of an inch; the specific gravity of lead is 11.4, and it will be found that the terminal velocity given by the formula is about 15 inches per second. Also, trying 6 inches for x, the expression $\epsilon^{-\frac{3}{8}\frac{\rho}{\sigma}\frac{x}{a}}=\epsilon^{-20}$ approximately, and therefore the terminal velocity is, at this depth, sensibly attained.

If the falling particle be imagined smaller, and the difference between ρ and σ be less than in the case we have tried, it is clear that the theoretical velocity will be very much diminished; and it should be noticed that in such cases the effect of fluid friction will become of great importance, and will materially diminish the velocity calculated from the preceding theory.

These results are illustrated by the well-known fact that minute particles of any kind subside in water with extreme slowness; for instance, the emery-powder used in polishing glass will sometimes take more than an hour to sink one foot.

Distribution of sediment in the sea. We can thus account for the great distances to which fine sand and mud are carried out into the sea by rivers and ocean currents, and for the evenness with which such sediment is deposited. In currents flowing at the rate of three or four miles an hour, small particles, sinking at the rate of two or three feet per second, may be carried out to distances exceeding a thousand miles, before they attain a depth of 800 feet, and it is easy to imagine the transportation of finer sediment over still greater distances*.

166. *An air-bubble ascends in fluid; it is required to determine its motion.*

The air-bubble as it ascends will expand; we shall suppose it always spherical and its size in any position determined by the pressure of the fluid around it, considered statically.

Let x be the height through which the bubble has ascended at any time and v its velocity, c its initial radius, and r its radius at the height x.

Then if $g\rho h$ be the initial pressure of the air and therefore of the fluid at its initial position, the pressure at the height x

$$= g\rho h - g\rho x,$$

and, by Marriotte's Law, $\dfrac{r^3}{a^3} = \dfrac{h}{h-x}$.

Also if σ' be the initial density and σ the density at the height x,

$$\frac{\sigma}{\sigma'} = \frac{a^3}{r^3} = \frac{h-x}{h},$$

the equation of motion is therefore

$$v\frac{dv}{dx} = g\frac{\rho-\sigma}{\sigma} - \frac{3}{16}\frac{\rho}{\sigma}\frac{v^2}{r},$$

* Lyell's *Principles of Geology*, Chap. XXI.

$$\text{or } \frac{dv^2}{du} + \frac{3}{8}\frac{\rho}{\sigma'}\frac{h^2}{(h-x)^2}\frac{v^2}{a} = 2g\left\{\frac{\rho}{\sigma'}\frac{h}{h-x} - 1\right\},$$

an equation from which v is to be determined.

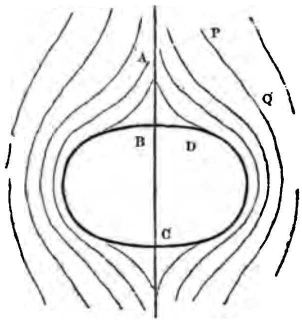
167. The preceding theory is confessedly imperfect, and, in fact, the results obtained from it are altogether discordant with the results of direct experiment. The following considerations may assist in shewing why this discord is to be anticipated.

Supposing, for simplicity, that no forces act on the fluid, the equation of steady motion gives

$$\frac{p-p'}{\rho} = \frac{1}{2}(u'^2 - u^2),$$

where p' and u' are the pressure and velocity at some known point, and p and u at any other point.

It appears from this equation that p depends on the velocity u and not on the direction of motion.



Suppose the stream to flow against a solid BC in the direction AB; the stream near AB will have its course diverted, and will flow in curved lines, PQ, &c., leaving a portion ADB at rest in contact with the surface, and these lines will be more nearly rectilinear and parallel to AB, the greater their distance from AB.

For the portion of surface BD, the pressure is therefore given by $p = p' + \frac{1}{2}\rho u'^2$, but, for the portion of surface beyond the point D, it cannot be assumed that the fluid is at rest, and the velocity along the tangential line of motion ought therefore to be calculated in order to find the pressure.

Moreover the pressure at the end C will probably be affected by the change of velocity of the fluid near the surface, and this has been entirely neglected: it is indeed possible that the increase of pressure at the end C, caused by the diminution of velocity, may counterbalance or even overcome the increased pressure at the end B, as in the experiment of Hachette referred to in Art. 142. An additional consideration is the effect of fluid friction, which may in such cases rise to importance. It appears from experiment that the tangential force varies nearly as the square of the velocity with which the fluid flows past the surface of a solid, when the velocity is not very small*.

EXAMPLES.

1. Compare the resistances on a sphere, and on a circular plate of the same radius immersed perpendicularly to the direction of a stream.

2. A solid cone moves in a fluid at rest in the direction of its axis, first with its vertex and secondly with its base foremost; compare the resistances in the two cases.

3. Find the resistance to a cycloidal lamina of given thickness moving in a fluid in the direction of its axis. Also find the resistance when the cycloid moves in the direction of its base.

4. A plane is moveable at right angles to a stream, the direction of which makes an angle α with the normal, and the force required to retain it at a constant velocity v in one direction is 4 times that required in the opposite direction. Shew that the velocity of the stream is $3v \tan\alpha$, or $\frac{1}{3}v \tan\alpha$, according as the forces in the two cases are in the same or in opposite directions.

5. The resistance on a cube moving in a fluid in the direction of its diagonal is to the resistance on the same cube moving perpendicular to its side :: $1 : \sqrt{3}$.

6. Given the base and height of the frustum of a cone moving through a fluid, find for what angle of the cone the resistance is a minimum.

* Professor Stokes, *On Fluid Friction.*

7. A solid formed by the revolution of a cycloid about its axis, moves in a fluid in direction of the axis; find the resistance of the fluid.

8. The curve $r = a(1 + \cos\theta)$ revolves round its axis and thus generates a solid of revolution. Find the resistance on this solid when it moves in a fluid, the direction of the motion of every point of the solid being parallel to the axis of revolution.

9. A cylinder on a circular base is cut by a plane parallel to its axis and bisecting its base: if the part cut off move horizontally through a fluid with its base and ends vertical, it will experience the same resistance, as would be exerted on an isosceles prism on the same base and with its equal sides inclined to one another at 120°, moving in the same manner.

10. Find the form of a solid of revolution, having a given circular base and a given height, such that the resistance upon it, when in motion through a fluid, in direction of its axis, may be a minimum.

CHAPTER XII.

THEORY OF SOUND.

168. THE theory of sound is included in the theory of the small oscillations of elastic fluids; that this is the case is evident from the consideration of a few experimental facts.

In the first place, the effect on the organs of the ear, called sound, is not produced unless there is an atmospheric communication between the ear and the disturbance causing the sound; if a bell is placed under a receiver, and the receiver exhausted as nearly as possible, the striking of the bell is not heard at all; moreover if the bell be struck during the process of exhaustion, the sound becomes gradually more faint as the exhaustion proceeds: it is evident therefore that the intensity of sound depends upon the density of the air, and diminishes with the diminution of that density.

169. That there is an actual motion of the atmosphere is shewn by the mechanical action which can sometimes be observed; for instance, glass windows are shaken by the firing of cannon, and, when the distance is small, are sometimes broken*; and similar effects may be produced by the sounds of an organ. It is well known that a musical note, sounded on any instrument, may produce a vibration, in unison with it, in some other body with which the instrument is not in contact; the human voice will, for example, set in motion a pianoforte wire, if the note sounded be in unison with the fundamental note of the wire, and this, it is evident, can only be effected by the transmission through the air of a mechanical action. Again, it is observed that, when sounds are heard through an atmosphere loaded with particles of dust, there is no sensible motion of the particles; and, in general, that sound is not necessarily accompanied by *wind*, unless the observer

* Windows have been broken at a distance of three or four hundred yards.

be near the origin of the sound: it follows from facts of this kind that sound is caused by *small* motions of the aerial fluid.

The Velocity of Sound.

170. It is a matter of very ordinary observation, that sound requires time for its propagation; a person standing near a cannon when it is fired, will hear the report almost at the same instant that the flash is visible to him; if, however, the cannon is at a distance, there will be a sensible interval between his perception of the flash and the report, and this interval increases with the distance.

It has been observed, moreover, that the velocity of sound is increased when the temperature is raised.

A great number of experiments have been made with the view of determining the velocity of sound, but from the various circumstances which affect its propagation, there are considerable discrepancies in the results obtained; Sir John Herschel considers that in dry air, at freezing temperature, the best approximation to the velocity of sound is about 1089 feet per second.

From experiments made by Arago and others in 1822, the velocity of sound when the barometer was at 29.8 inches, and the thermometer at 61°, was found to be 1118.4 feet per second.

171. *Sounds of different pitch and intensity travel with the same velocity.*

When a musical band is heard at a distance the harmony is unaffected, and it is therefore clear that there is no sensible difference in the periods of time required for the transit of the various notes produced at the same instant. This inference, however, can only be drawn for the limits of distance within which it is possible to hear the band at all; and it does not appear that direct experiments have been made for a greater distance than 951 metres, or about 1040 yards*.

* For an account of these various experiments, see Herschel's *Sound*, *Encyc. Metrop.* A piece of evidence may here be given, with reference to Art. (171) On a fine and still evening of June, 1858, the *Messiah* was performed in a *tent*, and the Hallelujah Chorus was distinctly heard, without loss of harmony, at a distance of two miles.

172. As it is through the air that sound is transmitted to the senses, the especial problem which offers itself, is the discussion, under various conditions, of the small vibrations of the aerial particles, but for a full consideration of the question, the laws of vibration of strings, of elastic rods and plates, of stretched surfaces, and of elastic solids require to be investigated.

These latter give rise to vibrations of the air, and the determination of the vari us modes in which their vibrations take place, forms, properly speaking, a part of the general question.

The investigation of the oscillatory movements of a solid body gives rise to equations of considerable complexity, and moreover the most important cases, those of musical sounds, depend in general upon the vibrations of strings, or rods, or of the air in cylindrical tubes; to these cases our attention will be confined.

Effect of Condensation on Temperature.

173. It is a well known fact that heat is produced by the sudden compression of air, and that, on the other hand, heat is lost by its sudden rarefaction; it follows therefore, in the small vibrations producing sound, in which the compressions and rarefactions take place very rapidly, that the air is rendered more elastic or less elastic, in a greater degree than is given by Marriotte's law.

These condensations being very small, we may consider without sensible error that the sudden changes of temperature are proportional to the condensations, a rarefaction being treated as a negative condensation.

If then a small portion of fluid, the density of which is D, be suddenly compressed so that its density is $D(1+s)$, s being a small quantity, the sudden change of temperature may be taken proportional to s and equal to γs, and the new pressure

$$= D(1+s)(1+\alpha\gamma s) = D\{1 + (\alpha\gamma + 1)s\},$$

neglecting the square of s, $= D(1+\beta s)$, if $\beta = \alpha\gamma + 1$.

174. Prop. *A hollow cylinder of indefinite length is filled with homogeneous air, a portion of which is disturbed in such a*

manner that all the particles in any section, perpendicular to the axis, are under the same initial circumstances of displacement; it is required to determine the resulting motion.

Let σ be the density of the air when undisturbed and $k\sigma$ its pressure. At the time t and at distance x measured parallel to the axis, let u be the velocity, p the pressure, and ρ the density.

Neglecting the action of gravity and supposing the surface of the cylinder perfectly smooth, the equation of motion is

$$dp = \rho \left\{ - \frac{Du}{Dt} \right\} dx,$$

$\frac{Du}{Dt}$ being the rate of change in the velocity of the particles of fluid, which at the time t, are at a distance x from the origin.

But $$\frac{Du}{Dt} = \frac{du}{dt} + u \frac{du}{dx}, \text{ as in Art. 116,}$$

observing that $\frac{du}{dt} \delta t$ is the change of velocity at the distance x during the time δt, $\frac{du}{dx} \delta x$ is the difference of velocities of the strata x and $x + \delta x$, at the same instant of time, and if $\delta x = u \delta t$, the sum of these two is the total change in the velocity of the particle referred to;

$$\therefore\ dp = \rho \left\{ - \frac{du}{dt} - u \frac{du}{dx} \right\} dx.$$

If $\rho = \sigma (1 + s)$, then, as in Art. (173), $p = k\sigma (1 + \beta s)$,

$$\text{and } \frac{dp}{\rho} = \frac{k\beta ds}{1 + s},$$

$$\text{or } \frac{k\beta ds}{1 + s} = \left(- \frac{du}{dt} - u \frac{du}{dx} \right) dx.$$

If ψ be a function of t such that $u = \frac{d\psi}{dx}$, we obtain, by integration with regard to x,

$$h\beta \log (1 + s) = F'(t) - \frac{d\psi}{dt} - \frac{1}{2} \left(\frac{d\psi}{dx} \right)^2.$$

The motion is supposed to be so small that the square of the velocity may be neglected, and therefore expanding log $(1+s)$,

$$k\beta s = -\frac{d}{dt}\{\psi - F'(t)\}.$$

Let $\psi - F(t) = \phi$; then $\frac{d\psi}{dx} = \frac{d\phi}{dx} = u$,

$$\text{and } k\beta s = -\frac{d\phi}{dt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The equation of continuity is

$$\frac{d\rho}{dt} + \frac{d(\rho u)}{dx} = 0,$$

or, substituting for ρ, and retaining only the first powers of small quantities,

$$\frac{ds}{dt} + \frac{d^2\phi}{dx^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Hence, from (1) and (2),

$$\frac{d^2\phi}{dt^2} = a^2 \frac{d^2\phi}{dx^2}, \text{ where } k\beta = a^2, \ldots\ldots\ldots\ldots (3),$$

is the equation which determines the oscillatory motions of the air in a straight tube.

The integral of this equation is

$$\phi = F_{,}(x + at) + f_{,}(x - at);$$

hence $$u = F(x + at) + f(x - at),$$

taking F and f as the derived functions of $F_{,}$ and $f_{,}$,

$$\text{and } s = -\frac{1}{a^2}\{aF(x + at) - af(x - at)\},$$

$$as = -F(x + at) + f(x - at).$$

The initial circumstance of motion will determine these functions.

Initially, when $t = 0$, let

$$u = \psi(x) \text{ and } s = \chi(x),$$

$$\text{then } F(x) + f(x) = \psi(x)$$

$$\text{and } F(x) - f(x) = -a\chi(x).$$

Hence $$\left.\begin{aligned} 2F(x) &= \psi(x) - a\chi(x) \\ 2f(x) &= \psi(x) + a\chi(x) \end{aligned}\right\} \quad\text{.............................}\quad (4);$$

$$\therefore\ 2u = \psi(x+at) - a\chi(x+at) + \psi(x-at) + a\chi(x-at),$$

$$2as = -\psi(x+at) + a\chi(x+at) + \psi(x-at) + a\chi(x-at);$$

and, if the functions $\psi(x)$ and $\chi(x)$ are given for all values of x from $-\infty$ to $+\infty$, the values of u and s are determined for all values of x and t.

In order to express the actual motion of a particular element of the fluid we must replace u by $\dfrac{dx}{dt}$, and obtain x in terms of t.

175. The equations of the preceding article may be also obtained without reference to the general equation of motion.

Let A be the area of a transverse section of the tube, x and $x+\delta x$ the distances from the origin of two particles near one another when at rest,

$$x+\xi,\ x+\delta x+\xi+\frac{d\xi}{dx}\delta x$$

the distances of the same particles when in motion at the time t.

Hence if σ be the density of the fluid in the space δx, and ρ of the same fluid when occupying the space $\delta x + \dfrac{d\xi}{dx}\delta x$,

$$\frac{\rho}{\sigma} = 1 + s = \frac{\delta x}{\left(1+\dfrac{d\xi}{dx}\right)\delta x} = 1 - \frac{d\xi}{dx} \text{ approximately,}$$

$$\text{or } s = -\frac{d\xi}{dx}.$$

If p be the pressure at the distance $x+\xi$, that is, the pressure at the time t about the particle whose distance when at rest is x, $p + \dfrac{dp}{dx}\delta x$ is the pressure at the distance $x + \delta x + \xi + \dfrac{d\xi}{dx}\delta x$.

The moving force on the mass $A\sigma\delta x = -A\dfrac{dp}{dx}\delta x$, and

$$\therefore\ A\sigma\delta x\frac{d^2\xi}{dt^2} = -A\frac{dp}{dx}\delta x.$$

But $$p = k\sigma(1+\beta s) = k\sigma\left(1 - \beta\frac{d\xi}{dx}\right),$$

$$\therefore\ \frac{dp}{dx} = -k\sigma\beta\frac{d^2\xi}{dx^2},$$

$$\text{and } \frac{d^2\xi}{dt^2} = k\beta\frac{d^2\xi}{dx^2} = a^2\frac{d^2\xi}{dx^2}.$$

The integral of this equation is of the form

$$\xi = \phi(x+at) + \psi(x-at),$$

$$\text{and } \therefore\ u = \frac{d\xi}{dt} = a\phi'(x+at) - a\psi'(x-at),$$

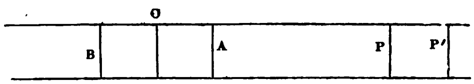
$$s = -\frac{d\xi}{dt} = -\phi'(x+at) - \psi'(x-at),$$

which are the same as the result of the preceding article if we write $F(x+at)$ for $a\phi'(x+at)$, and $f(x-at)$ for $-a\psi'(x-at)$.

176. *To find the velocity with which a disturbance is propagated along the tube.*

Let the initial disturbance extend through a space AB, (2λ), from $x = -\lambda$ to $x = +\lambda$; then $\psi(x)$ and $\chi(x)$ are each zero for all values of x, except those comprised between $x = \pm\lambda$, and, from the equations (4), it appears that $F(x)$ and $f(x)$ are subject to the same law.

First, consider the motion of the fluid at a point P, such that x, i.e. OP, is greater than λ.



In this case $x + at > \lambda$,

$$F(x+at) = 0,\quad f(x+at) = 0;$$

$$\therefore\ u = f(x-at),\quad as = f(x-at),\quad \text{and } u = as.$$

Also $f(x-at)$ is zero, and therefore u and s are zero, except for values of t which make

$$x - at < \lambda \text{ and } > -\lambda,$$

so that if τ, τ', be the times at which the motion of P begins and ends,

$$x - a\tau = \lambda,\quad x - a\tau' = -\lambda,\quad \text{and } \therefore\ \tau' - \tau = \frac{2\lambda}{a}.$$

Hence P is set in motion at the time τ, vibrates during the time $\frac{2\lambda}{a}$, and is afterwards at rest.

Again, if P_1 be another point of the tube, the fluid at P_1 will be set in motion at a time τ_1 such that

$$OP_1 - a\tau_1 = \lambda;$$

and, since $OP - a\tau = \lambda$, it follows that $\frac{PP_1}{\tau_1 - \tau} = a$; and therefore a is the rate at which the disturbance travels along the tube in the positive direction. When P has come to rest, the motion which P had at first will have been transmitted to a point at a distance 2λ from P, since $\frac{2\lambda}{a}$ is the time during which P is in motion.

The disturbance therefore travels along the tube in the form of a *wave* of constant length 2λ and with a constant velocity.

Secondly, consider a point on the negative side and such that $x < -\lambda$.

Hence $$x - at < -\lambda,$$

and therefore $$F(x - at) = 0, \quad f(x - at) = 0,$$

$$u = F(x + at), \; as = -F(x + at), \text{ and } u = -as.$$

These expressions will be zero unless

$$x + at > -\lambda \text{ and } < \lambda;$$

and therefore if x, x', be the distances of two points at which motion commences at the times τ, τ', respectively,

$$x + a\tau = -\lambda, \quad x' + a\tau' = -\lambda,$$

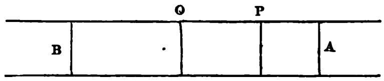
$$\text{and } x' - x = -a(\tau' - \tau),$$

from which it results, as before, that a is the velocity of propagation.

Similarly, the time of motion of any one element is $\frac{2\lambda}{a}$, and a wave is therefore propagated in the negative direction.

Lastly, consider the motion of a point between the limits AB of the initial disturbance.

The velocity and condensation are both given, {equations (4)}, by the sum of two functions, representing respectively the disturbances due to the two waves which we have shewn to be



travelling in opposite directions. By the principle of the superposition of small motions*, the motion is therefore the same as would be caused by the coexistence of two waves, travelling across the point P in opposite directions.

Now $\qquad OP + at < \lambda$, until $t = \dfrac{\lambda - OP}{a} = \dfrac{PA}{a}$,

after which $\qquad F(OP + at) = 0$;

and $\qquad OP - at > -\lambda$, until $t = +\dfrac{\lambda + OP}{a} = \dfrac{PB}{a}$,

after which $\qquad f(OP - at) = 0$;

that is, the motion of P is represented by the coexistence of two vibrations until the negative wave has travelled over a space AP, and it is then disturbed only by the positive wave travelling over BP.

177. It may be observed that of these two waves it is possible that only one may exist; if, for instance

$$\psi(x) - a\chi(x) = 0$$

for all values of x between $\pm\lambda$, then $F(x) = 0$ for *all* values of x, and only one wave is propagated.

The function $f(x - at)$ determines the wave which is propagated in the positive direction, and, for the continuance of this motion, the relation $u = as$ is necessary. If this relation be

* The principle of the superposition of small motions asserts that if a number of small disturbing causes act on a material particle, the resulting effect is *sensibly* the sum of the effect due to each cause acting singly.

Thus, if a quantity u would be changed by one disturbing cause acting alone into $u + \alpha u$, where α is very small; and by another into $u + \beta u$, the whole change, when the two act together, will be $\alpha u + \beta u$; for if the second cause act immediately after the first, the resulting additional change would be $\beta(u + \alpha u)$ or $\beta u + \alpha\beta u$, when $\alpha\beta u$ being a small quantity of the 2nd order may be neglected in comparison with $\alpha u + \beta u$.

destroyed, the wave so disturbed will give rise to two waves, one travelling in the positive and the other in the negative direction.

A similar remark is applicable to the function $F(x+at)$, which determines the wave travelling in the negative direction.

Definition of a Wave.

178. The term wave is applied to any state of motion transmitted through a substance the elements of which are slightly disturbed. The length of a wave is the distance between two consecutive surfaces of equal displacement, that is, two surfaces, the particles in which are, at the instant considered, in the same state of motion. In the case of the last article, a solitary wave, of the length 2λ, is propagated in the positive direction, and a similar wave in the negative direction, so that every portion of fluid in the tube has a phase of motion, and is afterwards at rest. Instances of the solitary wave may be seen in the effect of a gust of wind on a corn-field, or in the expanding circles produced by dropping a stone in still water.

If the original disturbance be repeated at the instant when the positive and negative waves, after traversing each other, have just cleared the space AB, and the process be continued, so that a series of waves follow closely on each other, then a particle of fluid, once set in motion, will vibrate isochronously, and if a number of points be taken at successive distances 2λ, the points of division will be the positions of all the particles of fluid which, at any one instant, are in the same state of vibration.

179. The nature of the original disturbance will determine the character of the wave, that is, the form, extent, and rapidity of the vibrations of which it consists.

Suppose, for instance, that the air in a tube is set in motion by the oscillations of a disc, the plane of which is perpendicular to the axis of the tube. When the disc moves in the direction OA, fig. Art. (176), the air on the side A is condensed, and on the other side is rarefied: if the motion of the disc then cease, a condensing wave will be propagated in the direction OA, and a rarefying wave in the direction OB, provided the separation

of the two portions of air by the disc be complete; but, when this separation does not exist or when vibrations can be transmitted through the disc, a complete wave, half condensing and half rarefying, will be propagated in each direction*.

If the disc make a complete oscillation, starting from rest and returning to its original position, complete waves will be propagated in each direction.

It should be observed that the range of vibration of the disc is not necessarily comparable with the length of the wave produced; the space through which the disc oscillates may be very small compared with the space AB. In fact, whatever be the extent of the disc oscillations, the wave AB depends only on the time of the oscillations, and on the velocity of propagation.

180. *The vibrations of the air in a tube closed at one end.*

Let P be the closed end. Then if a disturbance be excited over a space AB, (2λ), it will in general cause two waves travelling in opposite directions, and one will impinge on the fixed end P.

The analytical condition is that for all values of t the value of u is zero, and we have to determine the modification introduced by this condition into our previous results.

Let $OP = c$; then

$$F(c + at) + f(c - at) = 0,$$

for all values of t;

$$\begin{aligned} \therefore\ F(x + at) &= F\left\{c + a\left(t - \frac{c - x}{a}\right)\right\} \\ &= -f\left\{c - a\left(t - \frac{c - x}{a}\right)\right\} \\ &= -f(2c - x - at), \end{aligned}$$

and $$u = f(x - at) - f(2c - x - at),$$

observing that $f(z)$ is zero except for values of z between $\pm\lambda$.

Now consider the motion at Q, a section of the tube between O and P.

* The term wave is sometimes applied to either of these portions; each being the distance between points of zero velocity.

Let $x = OQ$, and first suppose $x < c - \lambda$. From $t = \dfrac{x+\lambda}{a}$, i. e. after the wave has passed over Q,

$$\overline{B \quad O \quad A \qquad\qquad Q \qquad\qquad Q' P}$$

$$f(x - at) = 0,$$

$$\text{and } f(2c - x - at) = 0 \text{ until } 2c - x - at = \lambda,$$

$$\text{or } t = \frac{2c - x - \lambda}{a} = \frac{c - \lambda + c - x}{a}$$

$$= \frac{AP + PQ}{a}.$$

$f(2c - x - at)$ is then finite, but again vanishes when

$$2c - x - at = -\lambda,$$

$$\text{or } t = \frac{2c - x + \lambda}{a} = \frac{AP + PQ + 2\lambda}{a};$$

and for all greater values of t is evanescent.

The motion of Q is therefore the same as if, when the front of the wave arrives at P, another wave, (2λ), were to start immediately, and travel in the opposite direction, following the negative wave at a distance $2AP$.

In other words the wave impinging on P is there reflected, and its motion exactly reversed.

If the distance of a point Q' from P be $< \lambda$, then for a certain time the motion of Q' will be the result of the superposition of two motions, namely, those due respectively to the incident and reflected waves: it will be caused by the reflected wave only when $t > \dfrac{BQ'}{a}$, and will cease altogether when $t > \dfrac{BP + PQ'}{a}$.

181. *The vibrations of the air in a tube open at one end.*

Let P be the open end at a distance c from O. The air in the tube at P being in immediate communication with the atmosphere, it may be assumed that its condensation is zero. This assumption is usually made for purposes of calculation, but it

appears from experiment that the point of zero condensation is at a little distance beyond the open end*.

We have then $s=0$ when $x=c$;

$$\text{or}\quad F(c+at)-f(c-at)=0 \text{ for all values of } t;$$

$$\therefore\; F(x+at)=F\left\{c+a\left(t-\frac{c-x}{a}\right)\right\}=f\left\{c-a\left(t-\frac{c-x}{a}\right)\right\}$$

$$=f(2c-x-at),$$

$$\text{and}\quad u=f(x-at)+f(2c-x-at).$$

By exactly the same reasoning as in Art. (180) it may be shewn that the wave on arriving at P is reflected and travels in the contrary direction with the same velocity.

182. *The vibrations in a tube of finite length.*

By the preceding investigations it appears that, if a disturbance be caused in a tube of finite length, the two waves, which start at first in opposite directions, will be both reflected on arriving at the respective ends of the tube, and their motions reversed, whether the ends of the tube are open or closed.

$P \qquad B \;\; A \qquad\qquad A' \;\; B' \qquad Q$

If PQ be the tube and AB the portion initially disturbed, the two waves will after reflection be superimposed at $A'B'$, which is at the same distance from Q that AB is from P; and the motions as thus described will recur continually.

The time of a complete *oscillation* of the two waves will be

$$\frac{AP+PQ+QA}{a},\ \text{or}\ \frac{2PQ}{a}.$$

If the initial disturbance extend over the whole of the tube, the motion of any portion of air in it will always be the result of the superposition of the two motions arising from the two waves propagated in opposite directions.

183. PROP. *To find the general equations for the vibrations of an elastic fluid.*

The process of forming these equations is almost identically

* Mr Hopkins, *On Aerial Vibrations. Camb. Phil. Trans.* 1838.

the same as that before employed for the simple case of rectilinear motion; we shall indicate it briefly.

If $\rho = \sigma(1 + s)$, and $p = \kappa\sigma(1 + \beta s)$,

the general equation of motion is

$$\int \frac{dp}{\rho} = -\frac{d\psi}{dt} - \frac{1}{2}(u^2 + v^2 + w^2) + F'(t),$$

where u, v, and w, are the component velocities, and ψ is a function such that

$$u = \frac{d\psi}{dx}, \quad v = \frac{d\psi}{dy}, \quad \text{and } w = \frac{d\psi}{dz},$$

assuming that $udx + vdy + wdz$ is a perfect differential.

Neglecting the squares of s and of the velocities, we obtain

$$\kappa\beta s = -\frac{d}{dt}\{\psi - F(t)\} = -\frac{d\phi}{dt}.$$

Also, substituting for ρ in the equation of continuity,

$$\frac{ds}{dt} + \frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0,$$

and therefore, $\dfrac{d^2\phi}{dt^2} = a^2\left(\dfrac{d^2\phi}{dx^2} + \dfrac{d^2\phi}{dy^2} + \dfrac{d^2\phi}{dz^2}\right)$,

where $a^2 = \kappa\beta$,

the equation which, with

$$u = \frac{d\phi}{dx}, \quad v = \frac{d\phi}{dy}, \quad \text{and } w = \frac{d\phi}{dz},$$

determines the motion.

184. *A disturbance is excited in a homogeneous atmosphere so as to proceed symmetrically from a centre; it is required to determine the motion.*

In other words the problem is, to determine the laws of the propagation of spherical atmospheric waves.

Taking the centre of the disturbance as the origin, the velocity (V) and condensation at any point will be functions of the distance (r) from the origin.

The equation of the previous article becomes

$$\frac{d^2\phi}{dt^2} = a^2\left(\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{r}\right),$$

$$\text{or, } \frac{d^2(r\phi)}{dt^2} = a^2\frac{d^2(r\phi)}{dr^2};$$

r and t being here independent variables.

Hence, $r\phi = F(r + at) + f(r - at)$,

and therefore, since $V = \dfrac{d\phi}{dr}$, and $a^2 s = -\dfrac{d\phi}{dt}$,

$$\left.\begin{aligned} V &= \frac{1}{r}\{F'(r+at) + f'(r-at)\} - \frac{1}{r^2}\{F(r+at) + f(r-at)\} \\ \text{and } as &= \frac{1}{r}\{f'(r-at) - F'(r+at)\} \end{aligned}\right\} \dots (\alpha).$$

In order to determine these functions, the initial values of V and s must be given for all values of r from 0 to ∞, and we must, besides, take account of the condition that, at the origin, $V = 0$ always, a condition obviously true, if the disturbance be symmetrical with respect to the centre.

Let $\psi(r)$ and $\chi(r)$ be the initial values of V and as, these functions being given for all values of r from 0 to ∞.

Then, $\psi(r) = \dfrac{d}{dr}\dfrac{F(r) + f(r)}{r}$, $\chi(r) = \dfrac{f'(r) - F'(r)}{r}$,

and therefore,

$$\left.\begin{aligned} \frac{1}{r}F(r) + \frac{1}{r}f(r) &= \int\psi(r) = \psi_1(r) + b, \\ f(r) - F(r) &= \int r\chi(r)\,dr = \chi_1(r) + c, \end{aligned}\right\} \dots (\beta),$$

b and c being arbitrary constants.

Suppose, if possible, that the expression for V, (α), can be expanded in powers of r; then, since V must vanish with r for all values of t, the terms involving negative powers of r must destroy each other, and this condition will be satisfied by assuming

$$F(r+at) + f(r-at) = Tr + T'r^2 + \dots$$

$$\text{and } F'(r+at) + f'(r-at) = T + T_1 r + \dots$$

From the first of these

$$F'(r+at)-f'(r-at)=\frac{r}{a}\frac{dT}{dt}+\dots$$

and therefore, making r evanescent, and putting z for at,

$$F(z)+f(-z)=0, \quad F'(z)-f'(-z)=0, \dots\dots\dots (\gamma),$$

for positive values of z.

By means of these equations it can be shewn that the constants in (β) will disappear in the final expressions for $F(z)$ and $f(z)$.

Thus, taking account of the constants only, we obtain from (β)

$$F(r)=\frac{1}{2}br-c, \quad f(r)=\frac{1}{2}br+c;$$

$$\therefore \; F(r+at)=\frac{1}{2}b\,(r+at)-\frac{1}{2}c, \quad F'(r+at)=\frac{1}{2}b,$$

and, if $r > at$,

$$f(r-at)=\frac{1}{2}b\,(r-at)+\frac{1}{2}c, \quad f'(r-at)=\frac{1}{2}b.$$

If $r < at$,

$$F(at-r)=\frac{1}{2}b\,(at-r)-\frac{1}{2}c, \quad F'(at-r)=\frac{1}{2}b,$$

but, from (γ), if $r < at$,

$$\begin{aligned} f(r-at) &= -F(at-r) \\ &= \frac{1}{2}b\,(r-at)+\frac{1}{2}c, \end{aligned}$$

the same as when $r > at$.

Substituting in (α), we find $V=0$ and $s=0$; the constants may therefore be omitted, and we obtain from (β), putting z for r,

$$\left.\begin{aligned} 2F(z) &= z\psi_1(z)-\chi_1(z), \\ 2f(z) &= z\psi_1(z)+\chi_1(z), \\ 2F'(z) &= \psi_1(z)+z\,\{\psi(z)-\chi(z)\}, \\ 2f'(z) &= \psi_1(z)+z\,\{\psi(z)+\chi(z)\}. \end{aligned}\right\} \dots\dots\dots\dots (\delta).$$

By these equations, if the initial disturbance be given, the subsequent motion is determined.

Poisson, *Mécanique*, Art. 660.

185. *Determination of the velocity of propagation of a spherical wave.*

Let the initial disturbance extend from $r = 0$ to $r = \alpha$; then $\psi(r)$ and $\chi(r)$ have given values from $r = 0$ to $r = \alpha$, and are zero from $r = \alpha$ to $r = \infty$. The integrals $\psi_1(r)$ and $\chi_1(r)$ are constant for all values of r greater than α, and, if we assume that $f(z)$ and $F(z)$ vanish when $z = \infty$, these integrals will vanish when $r = \infty$, and will therefore vanish for all values of r from 0 to ∞.

Hence, from the equations (δ), if $r < \alpha$, $F(r + at)$ and $F'(r + at)$ are finite as long as

$$t < \frac{\alpha - r}{a};$$

also $f(r - at)$ and $f'(r - at)$, are finite as long as

$$t < \frac{r}{a},$$

and, when $$t > \frac{r}{a},$$

we find from (γ) that $f(r - at)$ and $f'(r - at)$ are finite for values of t less than $\frac{\alpha + r}{a}$: it appears then that the original surface of disturbance is in motion during the time $\frac{2\alpha}{a}$.

If $r > \alpha$, $F(r + at)$ and $F'(r + at)$ are zero, and we have

$$V = \frac{1}{r} f'(r - at) - \frac{1}{r^2} f(r - at),$$

$$as = \frac{1}{r} f'(r - at).$$

But, from the equations (δ), $f(r - at)$ and $f'(r - at)$ are finite from

$$t = \frac{r - \alpha}{a} \text{ to } t = \frac{r + \alpha}{a},$$

and therefore the particles of fluid about the spherical surface of which r is the radius are in motion during the time $\frac{2\alpha}{a}$.

Moreover, the motion of a particle at the distance r commences when $t=\dfrac{r+a}{a}$, and of a particle at the distance r' when $t=\dfrac{r'+a}{a}$, and the motion extends from the sphere r to the sphere r' in the time $\dfrac{r'-r}{a}$; a therefore represents the velocity with which the wave motion is propagated.

At a considerable distance from the centre of the initial distance, the terms involving $\dfrac{1}{r^2}$ may be neglected, and we obtain

$$V=as,$$

the relation before obtained in the case of vibrations in a straight tube. This result might have been anticipated, for at a considerable distance from the centre a small portion of the wave front would be approximately plane, and would therefore follow the laws of motion of a plane wave.

186. *Nature of the motion when the initial displacement is small but not symmetrical with regard to a centre.*

In the case of a spherical wave, suppose a conical surface described, of very small vertical angle, with its vertex at the centre of the sphere; we may conceive the air within this cone to be isolated, without affecting its motion in any way.

Now, whatever be the form of the initial surface of displacement, we can suppose the aerial mass divided into a number of cones having their vertices at the origin, in each of which the velocity of propagation will be the same, and of the nature of the propagation of a spherical wave. After the lapse of a finite time, the several portions of the surface of disturbance, or wave surface, will be sensibly at the same distances from the origin, if the initial disturbance be of small extent, and therefore the wave surface will approximate, as it expands, to a spherical form.

187. *Intensity of Sound.*

From the equations of Art. (184), it appears that V diminishes with r; and, as the intensity of sound is measured,

in a given medium, by the square of V, it follows that at large distances from a centre of disturbance the intensity varies as $\frac{1}{r^2}$.

188. *Comparison with observation of the theoretical velocity of sound.*

Writing K for $1+\beta$, the expression for the velocity becomes

$$\sqrt{\left\{gh\frac{\sigma}{\rho}K(1+\alpha t)\right\}}.$$

Now taking a foot and a second as units of space and time, $g=32{\cdot}2$, and in dry air at the freezing temperature, the height of the barometer being 29·927 inches, the experiments of Biot give $\frac{\sigma}{\rho}=10463$.

The quantity K may be determined by observations on the increase of temperature in a given mass of air produced by a given condensation. From the experiments of Clement and Desormes the value obtained is 1·3492.

Hence, the velocity of sound at the freezing temperature

$$=\left\{32{\cdot}2\times 10463\times 1{\cdot}3492\times\frac{29{\cdot}927}{12}\right\}^{\frac{1}{2}},$$

which is approximately 1064 feet per second, and is less than the velocity, 1090 feet per second, given by observations*.

The discrepancy depends chiefly on the uncertainty of the value of K, as determined by direct observation; K is in fact best determined by equating the expression $\sqrt{\left(gh\frac{\sigma}{\rho}K\right)}$ to the observed velocity.

Specific Heat.

189. A physical meaning can be assigned to the coefficient K by introducing the idea of specific heat.

It is found that in order to produce a given change of temperature, different bodies require the application of different quantities of heat, and the relative specific heats of two bodies,

* For a list of the authorities on which this statement depends, see Herschel's *Sound*, Art. 16.

or of the same body in different states, are determined by comparing the quantities of heat required to produce the same change of temperature in equal masses of the two bodies.

For elastic fluids it is necessary to consider two cases; (1) when the pressure remains constant, the gas being allowed to expand, (2) when the volume is constant.

Let ρ be the density and p the pressure of a mass of air at a temperature (t), and suppose that by the application of the heat Q, the temperature is raised τ^0, the pressure remaining constant and the density being changed to ρ'; then

$$p = \kappa\rho\,(1 + \alpha t) = \kappa\rho'\,\{1 + \alpha(t+\tau)\}.$$

Now let the air be rapidly compressed into its original volume, and assume that the consequent increase of temperature (t') is proportional to the condensation.

Hence, $$t' = \mu s = \mu\left(\frac{\rho}{\rho'} - 1\right) = \frac{\mu\alpha\tau}{1+\alpha t},$$

and the whole increase of temperature due to the heat Q, when the volume is constant, is $\tau\left(1 + \frac{\mu\alpha}{1+\alpha t}\right)$; and if q be the heat required to produce a change τ of temperature in a mass of constant volume,

$$Q : q :: \tau\left(1 + \frac{\mu\alpha}{1+\alpha t}\right) : \tau;$$

or, $1 + \frac{\mu\alpha}{1+\alpha t}$ is the ratio of the specific heat of air under a constant pressure to its specific heat when its volume is constant.

It appears from experiment that this quantity is independent of t.

Suppose the air in which sound is propagated to have a temperature t, then, in the state of rest,

$$p = \kappa\rho\,(1 + \alpha t),$$

and in the state of motion,

$$p' = \kappa\rho\,(1+s)\,\{1 + \alpha\,(t + \mu s)\},$$

$$= \kappa\rho\,(1 + \alpha t)\left(1 + s + \frac{\mu\alpha s}{1+\alpha t}\right).$$

Comparing this with Art. (173), we obtain

$$1+\beta=1+\frac{\mu\alpha}{1+\alpha t}=K,$$

and the velocity of sound $=\sqrt{\left\{\frac{g\sigma h}{\rho}(1+\alpha t)K\right\}}$.

Propagation of Sound in Vapour.

190. The remark made in Art. 161, on the effect of a sudden compression of air in raising its temperature, enables us to explain the known fact that sound can be propagated through vapour in its state of greatest or saturating density.

The effect of a compression of the vapour, without change of temperature, would be the precipitation, in the form of dew, of some portion of it, but the compressions which are caused by rapid vibrations are always accompanied by an increase of temperature, the vapour remains uncondensed, and the vibrations are propagated through it in the same way as through the air.

191. The effect of *simultaneous disturbances from different centres* may be determined by observing that the equation,

$$\frac{d^2\phi}{dt^2}=a^2\left(\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}+\frac{d^2\phi}{dz^2}\right),$$

is satisfied if we take for ϕ an expression of the form

$$\frac{1}{r}\{F(r+at)+f(r-at)\}$$

$$+\frac{1}{r_1}\{F_1(r_1+at)+f_1(r_1-at)\}$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots$$

where $r, r_1, \ldots$ are the distances of a point in the fluid from the several centres of disturbance.

The condensation, $\frac{d\phi}{dt}$, is therefore the sum of the several partial condensations, and the velocities parallel to the axes at any point of the fluid, are given by the equations,

$$u = \frac{d\phi}{dx} = \frac{d\phi}{dr}\frac{dr}{dx} + \frac{d\phi}{dr_1}\frac{dr_1}{dx} + \dots$$

$$\text{or, } u = \frac{x}{r}\frac{d\phi}{dr} + \frac{x_1}{r_1}\frac{d\phi}{dr_1} + \dots$$

$$\text{and } v = \frac{y}{r}\frac{d\phi}{dr} + \frac{y_1}{r_1}\frac{d\phi}{dr_1} + \dots$$

$$w = \frac{z}{r}\frac{d\phi}{dr} + \frac{z_1}{r_1}\frac{d\phi}{dr_1} + \dots$$

where (x, y, z) (x_1, y_1, z_1) ... are the co-ordinates of the point referred to axes originating in the general centres.

The velocity in any direction is therefore the sum of the velocities in that direction due to the partial disturbances.

192. *Reflection of a spherical wave at a fixed plane.*

Suppose that two exactly similar spherical waves proceed from two centres; that is, let the velocities and condensations in the two waves be the same, simultaneously, at the same distances from the centres, and consider the nature of the disturbance which takes place at the plane which is equidistant from the two centres.

By the preceding article, it is clear that the resultant motions of the particles at this plane will be entirely parallel to the plane; it appears moreover, by the same reasoning, that the two waves will pass through each other, and afterwards proceed, as if each alone had been originally excited, and that, if a series of pairs of waves proceed from the two centres, the disturbance at any point will result from the combination, by the superposition of small motions, of the partial disturbances.

If now a rigid plane occupy the place of the geometrical plane equidistant from the centres, one of the centres of disturbance may be removed, without any other alteration in the circumstances of the motion, and, if the rigid plane be perfectly smooth, the velocities of the aerial particles in contact with it will be entirely parallel to it, and its action upon the spherical wave will be represented by a reflected wave following exactly the same laws of propagation as the incident wave.

The Refraction of Sound.

193. We have seen that a disturbance in an aerial column produces two waves travelling in opposite directions, and that in these two waves the conditions $v = as$, $v = -as$, are respectively satisfied.

If then anything occur to destroy the relation $v = \pm as$ in either wave, the effect will be the production of two new waves.

Suppose the aerial column to consist of two parts, containing different gases, and that one of the waves formed in one part impinges on the plane of separation, and thus produces a disturbance in the other part.

The states of motion of the two gases may be represented respectively by the systems of equations,

$$\left.\begin{aligned} v &= f(x - at) + F(x + at) \\ as &= f(x - at) - F(x + at) \end{aligned}\right\},$$

$$\left.\begin{aligned} v' &= \phi(x - a't) + \Phi(x + a't) \\ a's' &= \phi(x - a't) - \Phi(x + a't) \end{aligned}\right\};$$

observing that the state of motion to which f and F refer is that which exists after the commencement of the impulse of the wave.

At the plane of separation the two media must have the same motion and the same elasticity;

hence, if ρ, ρ', be the densities of the two media,

$$\kappa\rho(1 + \beta s) = \kappa'\rho'(1 + \beta' s'),$$

β and β' being constants depending on the heat developed by compression, and therefore, since in the position of equilibrium the pressures $\kappa\rho$, $\kappa'\rho'$ are equal,

$$\beta s = \beta' s'.$$

If then $x = l$ at the plane of separation, these two conditions give

$$f(l - at) + F(l + at) = \phi(l - a't) + \Phi(l + a't), \dots\dots\dots (1),$$

$$f(l - at) - F(l + at) = \mu\{\phi(l - a't) - \Phi(l + a't)\}, \dots\dots (2),$$

putting $\mu = \dfrac{\beta' a}{\beta a'}$.

Let $x = \pm a$ mark the range of initial disturbance; then v, v', s, and s' are all zero initially except for values of x between $\pm a$.

Hence, considering the second medium,

$$\phi(x) = 0, \quad \text{and} \quad \Phi(x) = 0,$$

and therefore $\Phi(x + a't) = 0$ always, from $x = l$ to $x = \infty$.

The relation $v' = a's'$ is therefore established, and a single wave is propagated.

But, considering the first medium, we obtain from (1) and (2), taking account of $\Phi(l + a't) = 0$,

$$as = \mu v,$$

and therefore, unless $\mu = 1$, there will be a reflected wave.

CHAPTER XIII.

MUSICAL SOUNDS.

194. ANY mechanical impulse of the air, of a sufficient degree of violence and suddenness, will produce a sound, and a series of impulses, following each other with sufficient rapidity, will produce the sensation of a continued sound. If the series of impulses are variable in their character, and follow no regular law of production, the result is a *noise*, but if the impulses are of the same kind, and produced at regular intervals, the result is a *musical note*.

A sound of such a nature is defined by three characteristics; these are, the *intensity* of the sound, which depends on the extent of vibration of the aerial particles, the *pitch* of the note, which depends on the rapidity with which the successive waves impinge on the ear, and a quality by which notes of the same intensity and pitch are distinguishable from each other, and which seems to be determined by the nature of the instruments employed in the production of the sound; the word *timbre* is sometimes used to express this quality*.

The velocity of propagation being the same for waves of any length, it will be seen that the pitch of a note is determined by the length of the wave, or by the time of vibration, and is higher or lower, as the time of vibration, or the length of the wave, is less or greater.

195. PROP. *To determine the notes which can be produced from a tube closed at one end.*

We may conceive a series of similar waves produced by the rapid oscillations of a disc in the column of air, the successive oscillations being exactly similar to each other.

* A further distinction is sometimes made by using the word *tone*. Thus the tone of a flute is different from that of other instruments, but the qualities of the notes obtained from different flutes may be different.

Suppose the disc to be at one end of the tube, the other end being closed, and that the motion of the air is steady, if such a motion be possible. By steady motion is here meant the perpetual recurrence, at any one point, of the same vibration.

Let the disc be at the origin, and take l for the length of the tube; then, since the velocity at the closed end is zero,

$$0 = F(l + at) + f(l - at) \text{ .... (1).}$$

Since the vibration of the disc is regular, the velocity at the origin may be represented by a periodic function $\phi(at)$, and

$$\therefore\ \phi(at) = F(at) + f(-at) \text{ .... (2).}$$

These two equations, if ϕ be given, determine F and f.

The equation (1) is true for all values of t, and therefore, putting $t - \frac{l}{a}$ for t,

$$F(at) = -f(2l - at),$$

$$\text{and } \phi(at) = f(-at) - f(2l - at),$$

a functional equation for the determination of f.

The function ϕ being periodic, we may hence infer that f is periodic, and that its period is the same as that of ϕ.

If λ be the length of a wave proceeding from a complete vibration of the disc, $\frac{\lambda}{a}$ is the period of ϕ, and therefore of f.

We have, generally,

$$v = F(x + at) + f(x - at);$$

$$\begin{aligned} \text{but } F(x + at) &= F\left\{l + a\left(t + \frac{x - l}{a}\right)\right\} \\ &= -f\left\{l - a\left(t + \frac{x - l}{a}\right)\right\} \text{ from (1),} \\ &= -f(2l - at - x), \end{aligned}$$

and therefore the points at which $v = 0$ are given by the equation

$$f(x - at) = f(2l - x - at).$$

Now f remains unchanged when t is changed by any multiple of $\frac{\lambda}{a}$;

$$\therefore\ x - at = 2l - x - at \pm m\lambda,$$

$$\text{or,}\ l - x = \pm m \frac{\lambda}{2}.$$

These points of zero velocity are called *nodes;* their distances from the closed end are $\frac{\lambda}{2}$, $2\frac{\lambda}{2}$, $3\frac{\lambda}{2}$, ..., and the distance between two consecutive nodes is half the length of a wave.

Assuming the oscillations of the disc to be exactly the same in both directions, the values of f will recur with opposite signs whenever at is changed by an odd multiple of $\frac{\lambda}{2}$.

$$\text{But, if}\qquad s = 0,\quad F(x + at) = f(x - at),$$

$$\text{or,}\ f(2l - at - x) = -f(x - at);$$

$$\therefore\ 2l - at - x = x - at \pm (2m + 1)\frac{\lambda}{2},$$

$$\text{or,}\ l - x = \pm (2m + 1)\frac{\lambda}{4}.$$

This gives a series of points of zero condensations, the distances of which from the closed end are $\frac{\lambda}{4}$, $3\frac{\lambda}{4}$, $5\frac{\lambda}{4}$, ...

These points are called *loops.*

If the length l of the tube were a multiple of $\frac{\lambda}{2}$, the origin would be a node, which is clearly impossible, and therefore the motion cannot be steady; if however the length be an odd multiple of $\frac{\lambda}{4}$, the origin will be a loop, and this is consistent with the circumstances of motion.

Taking the origin as a loop and the closed end as a node, it is evident that the greatest value of $\frac{\lambda}{4}$ is l, and therefore the vibration of longest period which can be kept up in the tube is

that for which $\lambda = 4l$. The sound thus produced is the fundamental note of the tube, or the lowest note which can be obtained from it, and the time of vibration for this note is $\frac{4l}{a}$.

A state of regular vibration is always possible when λ is such that

$$l = (2m + 1)\frac{\lambda}{4},$$

and therefore the times of vibration corresponding to the notes, placed in ascending order, which can be produced from the tube, are

$$\frac{4l}{a}, \frac{4l}{3a}, \frac{4l}{5a}, \ldots$$

being in the ratios $1 : \frac{1}{3} : \frac{1}{5} : \ldots$.

Reflection at the Disc.

196. Supposing that the vibrations of the disc are maintained, we have to consider its effect on the returning wave, and for this it is sufficient to remark that the motion would be practically very small compared with the rate of propagation of the aerial vibrations it excites, and the returning wave will be reflected by it as if it were fixed. The state of vibration of any particle will therefore result from the coexistence of a number of vibrations arising from the various waves which travel backwards and forwards in the tube and which are continually reinforced by new oscillations of the disc.

197. It is important to observe that the continuance of the sound depends on the fact that the tube is of finite length, and not merely upon the repetition of the impulses by the disc, the effects of which are to reinforce the fading intensities of the original vibration. In fact, a vibration once produced will be perpetually reflected at the ends of the tube, until it is destroyed by the friction of the tube, the imperfect elasticity of the closed end, or the friction of the air itself, so that if the disc were to make oscillations and then to be removed, the note would be produced but its intensity would rapidly diminish. As a matter

of fact such a note would not in general be heard at all after the cessation of the disturbance.

198. *Effect of discontinuous oscillations of the disc.*

Conceive the disc oscillations to take place at separated intervals; then, neglecting the faint echos arising from the continued reflections, a number of discrete waves would be propagated, and would produce on the ear, if of sufficient intensity, the sensation of a continued series of short cracking sounds. If, however, the intervals between the oscillations be made very small, so as to be inappreciable by the ear, the sensation will be that of a continued note.

This has been in several ways put to the test of experiment; for instance, in the Sirene of Cagniard de la Tour. In this instrument, the wind of a bellows is emitted through a small aperture close to which revolves a disc pierced with a number of holes arranged in a circle concentric with the axis of rotation; and, as the disc revolves, the intervals between the holes act as a cover and intercept the air. When the disc revolves with sufficient rapidity, the result is a continued note*.

199. We have supposed throughout the previous investigation that the force controlling the motion of the disc is sufficiently powerful to render the oscillations independent of the varying resistance offered by the air, and it has resulted that the steady motion requisite for the production of a musical note is not possible except for certain definite relations between the length of the tube and the time of oscillation of the disc.

In practice this is not the case, for, whatever mode of exciting a vibration be adopted, it is found that the action of the air within the tube modifies the original vibration, and that vibrations consistent with the possibility of steady motion are generally established, so that either the fundamental note or one of its harmonics is sounded.

200. PROP. *To determine the notes which can be produced from a tube open at one end.*

* In Herschel's *Sound*, Part II., an account will be found of experiments elucidating this point.

Suppose the vibrations excited by a disc at the other end, at which the origin is taken.

As in Art. (181), we have

$$F(l+at)=f(l-at),$$

and therefore

$$F(x+at)=F\{l+a(t+\frac{x}{a}-l)\}$$

$$=f\{l-a(t+\frac{x}{a}-l)\}$$

$$=f(2l-at-x),$$

$$\text{and}\quad as=f(x-at)-f(2l-at-x).$$

The function being periodic, if $s=0$, we have

$$2l-at-x=x-at\pm m\lambda,$$

and therefore $$l-x=\pm m\frac{\lambda}{2},$$

giving a series of loops at distances $m\frac{\lambda}{2}$ from the open end.

If then $l=m\frac{\lambda}{2}$, a series of notes can be obtained of which the times of vibration are

$$\frac{2l}{a},\ \frac{2l}{2a},\ \frac{2l}{3a},\ \frac{2l}{4a},\ldots.$$

It may be noticed that the time of vibration of the fundamental note of the open tube is half that of the fundamental note of a closed tube of the same length, and the note is therefore an octave higher.

201. Other notes than the harmonics just discussed can be obtained from tubes by making apertures at different points, and thus establishing communications with the external air. If, for instance, at a distance c from the end at which the disturbance is excited, an aperture be made of sufficient size, the air within the tube can only vibrate steadily when this aperture coincides with the position of a loop, and therefore $2c$ will be the longest possible wave, and $\frac{2c}{a}$ the time of vibration of the lowest

possible note. The other portion of the tube will be inoperative, unless indeed its length be a multiple of $2c$, in which case it might be anticipated that the air within it would vibrate in unison with the air in the length c, and thus perhaps increase the intensity of the sound.

By properly placing apertures, the notes of the diatonic scale, and their harmonics, can thus be produced from a single tube*.

The construction of a *flute* is an illustration of the preceding theory: it must be observed, however, that on account of the small size of the apertures and the difficulty referred to in Art. 181, the distances of the apertures from the ends are not exactly the same as would be given by the theory.

202. *Case of a tube closed at both ends.*

The effect of an aerial disturbance in a tube closed at both ends is given by the results of Arts. 180 and 182.

To find the notes producible from the tube, we must consider the conditions necessary for steady motion; and it is clear that for this purpose the two ends must be nodes, and that, if l be the whole length of the tube, and λ the length of the wave, l must be a multiple of $\frac{\lambda}{2}$. The times of the vibrations of the notes which can be produced are therefore

$$\frac{2l}{a}, \frac{2l}{2a}, \frac{2l}{2a}, \ldots$$

the same as for a tube open at one end.

We may suppose the disturbing cause to take effect at an opening, or *embouchure*, at the middle of the tube.

203. *Case of a tube open at both ends.*

The condition necessary for steady motion is that the two ends should be loops, and this case is therefore at once reduced to that of a tube open at one end and having the disturbance

* The ratios of the times of vibration corresponding to a set of notes in the diatonic scale are

$$1, \frac{8}{9}, \frac{4}{5}, \frac{3}{4}, \frac{2}{3}, \frac{3}{5}, \frac{8}{15}, \frac{1}{2},$$

ending at the octave of the first note.

excited at the other. If, however, the embouchure be in the middle of the tube the lowest note which can be obtained will be an octave higher than if the disturbance were excited at the end.

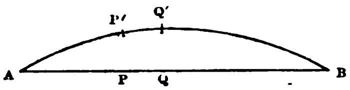
In fact, each of the two preceding cases, if the embouchure be in the middle of the tube, is equivalent to the combination of tubes, each of half the length of the tube considered; and it is easy to see that the two portions may vibrate in unison, and that, if the disturbance be excited at their plane of junction, they will do so.

204. The preceding investigations are applicable to the cases of tubes having a curved axis, provided the sectional area be not very large.

Organ pipes, for instance, may be bent or crooked in form, and it is found that the pitch of the note depends on the length of the axis of the tube, and is not affected by the form of the axis.

The Vibrations of Strings.

205. A piece of string or wire, tightly stretched between two fixed points, can be made to vibrate, and if the vibrations be sufficiently rapid, a musical note will be produced.



Let APB be a cord, stretched between the two points A, B, and represent by τ its tension in the position of rest: neglecting the curvature due to the weight of the cord, APB will be a straight line, and τ will be constant. We shall suppose the cord perfectly flexible and only slightly extensible.

Let $AP'Q'B$ be the position of the vibrating cord at the time t, $P'Q'$ being the position of the element PQ, and taking A for the origin, and AB for the axis of x, let $x+u$, y, z, be the co-ordinates of P', the position of P; u, y, z are therefore the displacements of P in directions of the axes, and are functions of x and t.

Let the tension at $P' = T$, $PQ = \delta x$, $P'Q' = \delta s$; then, resolving the tensions at P' and Q' parallel to the axes, their differences are respectively

$$\frac{d}{ds}\left\{T\frac{d(x+u)}{ds}\right\}\delta s,\quad \frac{d}{ds}\left(T\frac{dy}{dt}\right)\delta s,\quad \frac{d}{ds}\left(T\frac{dz}{ds}\right)\delta s.$$

Also, the increased extension being proportional to the increase of tension,

$$\frac{\delta s - \delta x}{\delta x} = \frac{T - \tau}{\mu},$$

μ being the modulus of elasticity of the string*.

Now $$\delta s^2 = (\delta x + \delta u)^2 + \delta y^2 + \delta z^2,$$

and, considering the small extensibility of the string, the angle made with AB by the tangent at any point will be always very small: hence the squares of $\frac{\delta y}{\delta s}$ and $\frac{\delta z}{\delta s}$ may be neglected, and we have

$$\delta s = \delta x + \delta u,$$

and therefore in the limit

$$T = \tau + \mu\frac{du}{dx}.$$

If l be the length and w the weight of the string, the mass of $PQ = \frac{w\delta x}{gl}$, and the equations of motion are

$$\frac{w\delta x}{gl}\frac{d^2u}{dt^2} = \frac{d}{ds}\left\{T\frac{d(x+u)}{ds}\right\}\delta s,$$

$$\frac{w\delta x}{gl}\frac{d^2y}{dt^2} = \frac{d}{ds}\left(T\frac{dy}{ds}\right)\delta s,$$

$$\frac{w\delta x}{gl}\frac{d^2z}{dt^2} = \frac{d}{ds}\left(T\frac{dz}{ds}\right)\delta s.$$

Putting $$\frac{gl\mu}{w} = c^2 \text{ and } \frac{gl\tau}{w} = a^2,$$

* If the string be very slightly extensible, μ is large, and is conveniently determined by the extension produced when a given weight is supported by a given length of string.

taking account of the above relations, and neglecting the products

$$\frac{du}{dx}\frac{dy}{ds}, \quad \frac{du}{dx}\frac{dz}{ds},$$

these equations become

$$\frac{d^2u}{dt^2}=c^2\frac{d^2u}{dx^2}, \quad \frac{d^2y}{dt^2}=a^2\frac{d^2y}{dx^2}, \quad \frac{d^2z}{dt^2}=a^2\frac{d^2z}{dx^2}.$$

The variables being separated, it appears that the three sets of vibrations are independent of each other, and the motion therefore results from the co-existence of the three vibrations.

206. These equations can be treated as in previous Articles, and it appears that c and a are respectively the velocities of propagation of longitudinal and transversal vibrations, the former being independent of the tension, and the latter depending on the tension, but not on the extensibility of the cord.

Let b be the length of the cord, of which the weight is τ, then $\frac{\tau}{w}=\frac{b}{l}$, and $a^2=bg$, that is, *the velocity of propagation of transversal vibrations is the velocity which would be acquired by a heavy body falling through half the length of a portion of the cord, of which the weight is equal to the tension.*

The equations for y and z being the same, it follows that, if the original displacement be in a plane through AB, the motion will be always in that plane.

207. *Reflection.* It may be shewn, exactly as in the case of aerial vibrations, that any disturbance of the cord will produce two waves, travelling in opposite directions, and continually reflected at the fixed end of the tube.

Nodes and ventral segments. Assuming the motion in one plane, the plane xy, the motion is determined by the equation

$$y=F(x+at)+f(x-at),$$

and since, at the points A and B, $y=0$, and $\frac{dy}{dt}=0$, we have, for all values of t,

$$0=F(at)+f(-at),$$
$$0=F(l+at)+f(l-at);$$

$$\therefore\ F(x+at) = F\left\{l + a\left(t - \frac{l-x}{a}\right)\right\}$$

$$= -f\left\{l - a\left(t - \frac{l-x}{a}\right)\right\}$$

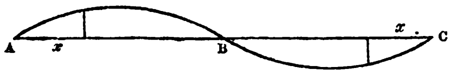
$$= -f(2l - x - at),$$

and $$y = f(x - at) - f(2l - x - at),$$

or, if $x' = 2l - x$,

$$-y = f(x' - at) - f(2l - x' - at).$$

It is inferred from this equation that if the string were continued beyond B, the displacement of a portion l would be always the same as that of AB, but in the opposite direction, and the point B would remain at rest.



The curve may evidently be continued above and below the line, and it follows therefore that, if the cord be divided into any number of parts of equal length, regular recurrence of the same vibrations may exist in each part, and the points of division remain at rest.

These points are nodes, and the portions between them are called ventral segments.

208. *Harmonics.* The time of a complete oscillation of the whole string is $\frac{2l}{a}$, but if it be divided into ventral segments of the length $\frac{\lambda}{2}$, all the portions will oscillate simultaneously in the time $\frac{\lambda}{a}$, and the note produced will depend on λ.

The harmonics of the string are therefore given by the equation

$$n\frac{\lambda}{2} = l,$$

and the times of vibrations of the notes are

$$\frac{2l}{a},\ \frac{2l}{2a},\ \frac{2l}{3a},\ldots\ldots\ldots$$

209. *Coexistence of harmonics.* The functions F and f being arbitrary, the equation for y may be written

$$y = F_1(x+at) + F_2(x+at) + \ldots + f_1(x-at) + f_2(x-at) + \ldots.$$

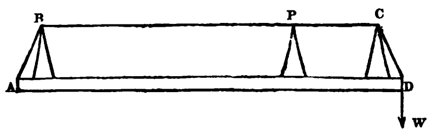
or $$y = y_1 + y_2 + \ldots\ldots$$

if $y_1, y_2, \ldots$ be vibrations represented by the functions $F_1, f_1, F_2, f_2, \ldots$ and therefore two or more vibrations of different kinds may coexist.

In practical confirmation of this result it is well known, that, besides the fundamental note of a stretched cord or wire, several of its harmonics may be heard at the same time, or indeed any number of the harmonics if the vibrations have sufficient intensity.

The Monochord.

210. The results of theory may be tested by this instrument, which in its simplest form consists of a piece of wire or catgut fastened at one end, and stretched over two fixed edges B and C, fixed to a sounding-board, by a weight at the other



end. Between the points B, C, is a moveable bridge, by means of which any point of the string can be reduced to rest, and therefore by varying the weight and the position of the bridge any note can be produced.

This apparatus may be employed to determine the rates of vibration of musical notes. Thus, if the bridge be moved until the fundamental note of BP is the same as any particular note, that is, in unison with it, and if $BP = l$, the time of vibration

$$= 2\sqrt{\left(\frac{lw}{g\tau}\right)},$$

where w is the weight of l, and τ the tension. The length l can be obtained from a graduated scale on AD, and τ is the weight suspended to the string.

For the exemplification of the theory of harmonics it is convenient to have two wires of the same substance, and of the same length, fastened to the sounding-board, and for this purpose

it is not necessary to produce the tension by means of a weight. The wires may be tightened by screws, and equality of tension can be secured by sounding their fundamental notes. The moveable bridge may be so constructed as to be in contact with one wire and not with the other.

Longitudinal Vibrations of Rods.

211. Suppose that vibrations are excited in a straight rod which is slightly compressible and slightly extensible in the direction of its axis, and that the motion of every particle in the same normal section is the same and in direction of the axis.

Let AB be the axis of the rod, the end A being fixed. Taking A for the origin, let x be the distance of a particular section P from A when undisturbed, and $x+u$ its distance AP' at the time t, $PQ=\delta x$, the length of an element, Q' the position of Q at the time t, T the tension, or resistance to compression, at P', and $T+\delta T$ at Q'.

The equation of motion is therefore

$$\frac{w\delta x}{gl}\frac{d^2u}{dt^2}=\delta T,$$

w being the weight and l the length of the rod.

Let αa be the extension, or compression, of a length a of the rod produced by a force W, and assume that the law of extension is the same as that of elastic strings. This assumption, if not true within the extreme limits of possible extension or compression, is justifiable for the small extensions which take place in vibrations, and we have, since δu is the extension of δx,

$$\frac{\delta u}{\alpha a}=\frac{\delta x}{a}\frac{T}{W}, \quad \text{or} \quad \frac{T}{W}=\frac{1}{\alpha}\frac{du}{dx},$$

and

$$\frac{d^2u}{dt^2}=\frac{lg}{\alpha}\frac{W}{w}\frac{d^2u}{dx^2}.$$

This equation is the same as that which defines the longitudinal vibrations of elastic strings, and might have been deduced from it by neglecting transversal motions; also, putting

$$\frac{lgW}{\alpha w}=c^2,$$

it appears that c is the velocity of propagation.

If both ends of the rod are fixed, we have $u=0$, and $\frac{du}{dt}=0$, when $x = 0$, or l, and, comparing tension with condensation there is an exact analogy between this case and that of a column of air in a tube closed at both ends. If one end is free,

$$T=0, \text{ or } \frac{du}{dx}=0,$$

when $x=l$, and the analogous case is that of air in a tube open at one end.

The times of vibration of the fundamental notes in these two cases are therefore $\frac{2l}{c}$ and $\frac{4l}{c}$ respectively.

The Propagation of Sound through Liquids.

212. Liquids, it is well known, are not absolutely incompressible, but, as very great force is required to produce a sensible compression, it is sufficient in all ordinary cases to neglect the change produced in the volume of a compressed liquid.

The great elasticity of water and other liquids renders however such media more capable of transmitting small vibratory motions than the air, and the velocity of propagation is in fact more than four times that of sound in air.

Suppose vibrations propagated along a column of water confined in a straight tube; substituting the compressibility of water for that of the rod, the case is the same as that of Art. (211), and therefore if αa be the compression of a length a of the column produced by a force W, and w the weight of a length l of the column, the velocity of propagation is given by the expression $\left(\frac{lgW}{\alpha w}\right)^{\frac{1}{2}}$.

213. It has been found that an additional pressure of one atmosphere produces in water, at the freezing temperature, a compression given by $\alpha = \cdot 000049589$; that is, αl is the diminution produced in the height of a column l of water by the weight of a column of mercury 29·927 inches in height.

If ρ be the density of mercury, σ that of water, and if κ be the sectional area of the column,

$$w = g\sigma l\kappa, \text{ and } W = \frac{1}{12} g\rho\kappa (29{\cdot}927);$$

and the velocity of sound in water $= \left\{\frac{1}{12} \frac{g\rho}{\sigma\alpha} (29{\cdot}927)\right\}^{\frac{1}{2}}$

$$= \left(\frac{32{\cdot}2 \times 29{\cdot}927 \times 13{\cdot}595}{12 \times {\cdot}00004958}\right)^{\frac{1}{2}},$$

since, at the freezing temperature, $\frac{\rho}{\sigma} = 13{\cdot}595$.

Calculating by logarithmic tables the values of this expression, we obtain, as the velocity of sound in water, 4693 feet per second.

By experiments made in the Lake of Geneva in 1826, the velocity of sound was found to be 4708 feet per second, the temperature of the water being about $8^{\circ} C$.

The compressibility of the water, at the freezing point, was found to be the same as at the temperature $8^{\circ} C$, and therefore the quantity α may be considered as unaffected by a change of temperature. Moreover, since the atmospheric pressure employed is that of a *standard* atmosphere, the quantity $\rho\,(29{\cdot}927)$ is also unaffected by a change of temperature.

The only element then which can vary in the expression for the velocity is the density of water, and, as this density is a maximum for a temperature of about $4^{\circ} C$, it may be anticipated that the densities at 0° and at $8^{\circ} C$ will be very nearly the same.

Such in effect is the case, the densities of water at 0°, 4°, and 8°, being, according to the results of Despretz*, in the ratios ·999873 : 1 : ·999878.

The velocity of sound should therefore be very nearly the same at $8^{\circ} C$ as at $0^{\circ} C$, and we must look for the causes of the discrepancy noticed above in the presence of extraneous substances in the water, and in the numerous errors to which observations of such a kind are necessarily liable.

It may be noticed that the heat developed by compression does not appear to affect in a sensible degree the velocity of sound in water.

* Dixon's *Treatise on Heat*, page 73.

214. When two strings of the same kind, very nearly in unison with each other, are set in vibration together, an intermitting sound is produced, and the alternations of intensity follow each other at regular intervals. If, for instance, the two strings belonging to any note of a pianoforte are not quite in unison, the note heard is alternately loud and faint*. Such alternations of intensity are called *beats*, and the more nearly the strings are in unison the greater is the interval between the beats.

Let τ, τ', be the times of vibration in fractions of a second of two strings, and suppose the vibrations to commence in the same plane; at first, the two vibrations will reinforce each other, but the faster string will gain on the other, until the vibrations are in opposite phases, in which case the two vibrations will partially destroy each other, and if the strings are exactly alike, and nearly in unison there will be an instant of almost perfect silence, after which the vibrations will again gradually reinforce each other.

Let the faster gain one vibration in x seconds; then

$$x\left(\frac{1}{\tau}-\frac{1}{\tau'}\right)=1, \quad \text{or } x=\frac{\tau\tau'}{\tau'-\tau},$$

which is the period of the beats, and is evidently greater, as $\tau'-\tau$ is smaller in comparison with τ or τ'.

215. It is not essential to the production of beats that the two strings should be nearly in unison; beats will also be heard when two strings form very nearly a concord. Suppose for instance in the case of a perfect fifth, in which the ratio of the vibrations should be 3 : 2, that one string makes 201 vibrations while the other makes 300; then about the 100th of the first or the 150th of the second, the former will have gained half a vibration on the other, and the two will be opposed.

Beats will result which are very distinctly marked; and in a similar manner beats can be obtained from other concords.

The earliest notice of these sounds is by Sauveur, about

* If there are three strings to a note, as is frequently the case, there will be a triple series of beats, arising from each pair of strings.

1700. Their theory is given in Smith's *Harmonics*, a treatise published in 1749.

216. Of a different nature are the *resultant sounds* which are sometimes heard when the concord of two notes is perfect. In the case of a perfect fifth every second vibration of one coincides with every third, and the effect produced is that of a note exactly one octave below the lowest note of the concord.

These sounds, called Tartini's beats, are discussed in a treatise, by Tartini, dated 1754. The term *subharmonics* has been lately applied to them by musical writers.

In general, if in a certain fraction (τ) of a second, one note makes m vibrations and the other n, the period of vibration of the resultant subharmonic is τ, m and n being supposed prime to each other.

217. *Limits of audibility.* Any aerial disturbance of sufficient intensity will produce a sound of some kind, but for the production of a note or continuous sound, it is necessary that periodical vibrations should recur with a certain degree of rapidity. It is stated by writers on music that when the number of vibrations is less than 16 per second, the successive impulses are separately appreciated, and the sensation of a continuous sound therefore implies that the number is greater than 16 per second.

On the other hand, if the number of vibrations be greatly increased, that is, if the pitch of the note be very much raised the sound becomes gradually faint, and beyond a certain limit is quite lost. This limit varies for different persons, but the general range of human hearing from the lowest note of an organ to the highest appreciable sound appears to be about nine octaves.

Hence the times of vibration of the extreme notes, and therefore the lengths of the corresponding waves, are approximately in the ratio $2^9 : 1$.

If a be the velocity of sound, and l the length of an organ-pipe, its fundamental note arises from vibrations of which the period is $\frac{2l}{a}$, and, equating this to $\frac{1}{16}$,

$$\text{we obtain } l = \frac{a}{32} = \frac{1090}{32} = 34 \text{ feet nearly,}$$

and the wave length is therefore about 68 feet.

Hence the shortest wave length is about $\frac{68}{2^9}$ feet or 1.6 inches.

From a series of experiments performed by Savart, it appears that this range may be extended, and that the limits of sensibility of the ear are frequently separated by eleven octaves.

In taking 1090 feet per second as the velocity of sound, we have supposed that the temperature is near the freezing point; if the temperature be greater the velocity is greater, and the length l may therefore be increased, or, if l be given, the time of vibration will be diminished. This is in accordance with the known fact that the pitch of an ordinary open organ-pipe is raised by an increase of temperature.

APPENDIX.

1. Some of the results of Articles 25 and 26 may be obtained in a more elementary manner, as in the following propositions.

When fluid is at rest under the action of gravity the pressure is the same at all points in the same horizontal plane.

Take any two points P, Q, in the same horizontal plane, describe about the straight line PQ a very thin prism bounded by vertical planes perpendicular to PQ, and suppose the fluid within this prism to be solidified. The solid PQ is kept in equilibrium by the fluid pressures on its surface and by its weight, and since the pressures on the ends are parallel to PQ, and on the rest of the surface perpendicular to PQ, it follows that the pressures on the ends balance each other. Now taking κ as the sectional area, and p, q, as the measures of the pressures at P, and Q, the pressures on the ends when k is very small are $p\kappa$ and $q\kappa$, and therefore $p = q$.

This reasoning applies to the case of heterogeneous fluids.

2. *To find the pressure at any given depth below the surface of an incompressible fluid.*

Suppose a thin vertical prism of the fluid AP, extending from the surface at A to the point P, to be solidified. Then the prism AP is kept at rest by the horizontal pressure of the fluid on its surface, by its own weight, and by the fluid pressure upwards on the end P, and the pressure of the air on the surface at A.

Therefore the pressure on the end $P =$ the weight of the prism + the pressure on the end A.

Let the fluid be homogeneous, its density being ρ; also let $AP = z$, k = the area of the base of the prism, supposed very small, Π = the pressure of the air, and p = pressure at P;

$$\therefore\ p\kappa = g\rho\kappa z + \Pi\kappa, \text{ or } p = g\rho z + \Pi.$$

It should be noticed that, whatever be the shape of the upper end A, the resultant vertical pressure of the air $= \Pi\kappa$.

3. *The surface of a fluid at rest is a horizontal plane.*

For, taking a fixed horizontal plane in the fluid, the pressure at any point of which is p, and taking Π constant, z must be constant, and the surface is therefore a plane parallel to the fixed horizontal plane.

4. *The common surface of two fluids that do not mix is a horizontal plane.*

Draw a vertical line ABC, meeting the upper surface in A, the common surface in B, and a fixed horizontal plane below the common surface in C.

Let ρ, ρ', be the densities of the upper and lower fluids respectively, $AB = z$, $BC = z'$.

Then the pressure at $C = g\rho' x'$ + pressure at B,

$$\text{or, } p = g\rho' z' + g\rho z,$$

where p, by Art. (1), is constant for all points in the horizontal plane through C.

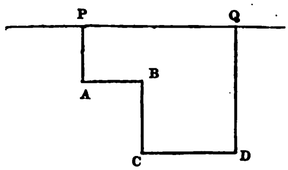
Also, the upper surface being horizontal, AC is constant, and we have, if $AC = d$,

$$z + z' = d, \text{ and } \rho z + \rho' z' = \frac{p}{g};$$

it follows therefore that z and z' are constant, and the common surface is parallel to the upper and lower surfaces.

5. The proposition of Article (1) is true when the straight line PQ does not lie wholly within the fluid. For we can reduce Art. (2) to any such case, by supposing any portion of the fluid between P and Q to be made solid, and such a supposition will not affect the pressures at P and Q.

Or we may suppose P and Q connected by a number of vertical and horizontal lines, and reason thus, taking the figure to represent a particular case.



$$\begin{aligned}\text{Pressure at } C &= \text{Pressure at } B + g\rho BC,\\ &= \text{Pressure at } A + g\rho BC,\\ &= g\rho(AP + BC) + \text{Pressure at } P.\end{aligned}$$

$$\text{Pressure at } D = g\rho QD + \text{Pressure at } Q.$$

But the pressures at C and D are equal, and $QD = AP + BC$; therefore the pressures at P and Q are equal.

Similar remarks may be applied to Article (4), when the vertical line AP does not lie wholly in the fluid.

6. *If two fluids, which do not mix, meet in a bent tube, or in any way communicate with each other, the heights of the free surfaces above the common surface are inversely as the densities.*

For their pressures at the common surface are the same, and if z, z', be the heights of the upper surfaces above the common surface, and ρ, ρ', the densities, these pressures are respectively

$$g\rho z + \Pi, \text{ and } g\rho' z' + \Pi,$$

and equating these expressions, we obtain, $\dfrac{z}{z'} = \dfrac{\rho'}{\rho}$.

7. *To find the whole pressure of a homogeneous inelastic fluid at rest under the action of gravity on any surface.*

Suppose the surface to be the limit of a polyhedron formed of small plane areas a_1, a_2, ..., and let z_1, z_2, ... be the depths below the surface of the centres of gravity of these planes.

Let a_1, a_2, ... be very small, so that the pressures at all points of any one area may be considered to be the same.

By Art. (2), taking the surface pressure Π to be zero, these pressures are respectively $g\rho z_1$, $g\rho z_2$, ...;

$$\therefore \text{ the whole pressure} = g\rho z_1 a_1 + g\rho z_2 a_2 + \dots$$
$$= \Sigma(g\rho a z) = g\rho\Sigma(az).$$

But, if $\bar{z}$ be the depth of the centre of gravity of the surface,

$$\bar{z}\ \Sigma(a) = \Sigma(az),$$

and therefore the whole pressure $= g\rho\bar{z}\ \Sigma(a)$.

Hence, in the limit, putting S for the whole surface, $\Sigma(a) = S$, and the whole pressure $= g\rho z S$.

In assuming that the pressure is uniform over each area, an error is introduced in each of the expressions $g\rho z a$.

If, however, $a(g\rho z + \varpi)$ be the pressure on an element a, ϖ is less than the greatest difference between the pressures at any two points of a, and therefore, when a is indefinitely diminished, ϖ vanishes in comparison with $g\rho z$, and consequently, $\Sigma(\varpi a)$ vanishes compared with $\Sigma(g\rho z a)$, and the whole pressure $= \Sigma(\varpi a + g\rho z a) = g\rho\bar{z}S$.

Virtual Velocities.

8. It may be shewn that the principle of virtual velocities is true in the case of a mass of incompressible fluid at rest, and not acted upon by any extraneous force.

Suppose the fluid in a closed vessel, having a number of apertures in which cylinders containing moveable pistons are fitted, and that equilibrium exists under the action of forces P, P', ... applied to these pistons.

Let κ, κ', ... be the areas of the pistons; then, since there is no extraneous force in action, the pressure at all points of the fluid is the same, and therefore, if p be the pressure,

$$P = p\kappa, \quad P' = p\kappa', \quad \dots\dots\dots\dots\dots\dots \quad (1).$$

Let the pistons be moved through spaces x, x', ... and suppose the new position one of equilibrium, so that the equations (1) still hold.

The volume being unchanged, we must have

$$K\kappa + K'\kappa' + \ldots = 0,$$

and therefore $P\kappa + P'\kappa' + \ldots = 0,$

which is the equation of virtual velocities.

9. *In a mass of fluid, revolving uniformly about a vertical axis, a very small sphere, of greater density than the fluid, is immersed, and is attached by a string to a point in the axis; it is required to find the positions of equilibrium of the sphere relatively to the fluid.*

One position of equilibrium is clearly that in which the string is vertical, but the sphere may also rest with the string inclined at some angle (θ) to the vertical. In this position the forces acting on the sphere are the fluid pressure, the weight of the sphere, and the tension (t) of the string.

To find the fluid pressures, imagine the sphere removed, and its place occupied by a solidified sphere of the fluid, and let V be the volume of the sphere, r its distance from the axis, and ρ the density of the fluid.

The horizontal and vertical resultants of the fluid pressures are therefore $V\rho\omega^2 r$ towards the axis, and $V\rho g$ upwards, since the weight of the solidified fluid is supported, and its circular motion maintained, by the fluid pressures.

Hence, we must have, if ρ' be the density of the sphere,

$$t \sin\theta + V\rho\omega^2 r = V\rho'\omega^2 r,$$

$$t \cos\theta + V\rho g = V\rho' g;$$

$$\therefore \tan\theta = \frac{\omega^2 r}{g}, \text{ and } t = V(\rho' - \rho)\sqrt{\omega^4 r^2 + g^2}.$$

The position of equilibrium is therefore the same as if the sphere and string formed a conical pendulum, the only effect of the fluid being a diminution of the tension.

10. *Remark on the example. Art.* 31. In order to render this case more easily conceivable, we may suppose, as an instance, that the fluid is contained in a wedge of a cylinder formed by passing two planes through its axis, and that the wedge revolves round the axis.

Taking the upper end as the plane xy, and measuring z along the axis downwards, we have

$$\frac{p}{\rho} = \frac{1}{2}\omega^2(x^2 + y^2) - gx\sin\alpha + gz\cos\alpha + C,$$

and at the upper end, where $z = 0$,

$$\frac{p}{\rho} = \frac{1}{2}\omega^2\left(x - \frac{g\sin\alpha}{\omega^2}\right)^2 + \frac{1}{2}\omega^2y^2 + C - \frac{g^2\sin^2\alpha}{2\omega^2}.$$

The lines of equal pressure are therefore circles having their centre at the point $\left(\frac{g\sin\alpha}{\omega^2}, 0\right)$, and if the vessel be only just filled, we may assume that at this point the pressure vanishes, and $\therefore\ C = \frac{g^2\sin^2\alpha}{2\omega^2}$.

11. PROP. *A mass of air being suddenly compressed or dilated, it is required to find the change of pressure.*

We shall assume, as in Art. (173), that, for a small condensation, the increase of temperature is proportional to the amount of condensation.

Let p, ρ, t be the original pressure, density and temperature, and let δt be the change in t due to a sudden change $\delta\rho$ in ρ; then

$$p = k\rho\,(1 + \alpha t), \text{ and } \delta t = \mu\frac{\delta\rho}{\rho} \text{ or } \frac{dt}{d\rho} = \frac{\mu}{\rho};$$

$$\therefore\ \frac{dp}{d\rho} = k\,(1 + \alpha t) + k\mu\alpha,$$

$$\frac{1}{p}\frac{dp}{d\rho} = \frac{1}{\rho}\left(1 + \frac{\mu\alpha}{1 + \alpha t}\right) = \frac{K}{\rho},$$

whence

$$\frac{p'}{p} = \left(\frac{\rho'}{\rho}\right)^K,$$

taking p' and ρ' as the new pressure and density.

But, if τ be the total change of temperature,

$$\frac{p'}{p} = \frac{\rho'\{1 + \alpha(t + \tau)\}}{\rho\,(1 + \alpha t)};$$

$$\therefore\ \frac{1 + \alpha(t + \tau)}{1 + \alpha t} = \left(\frac{\rho'}{\rho}\right)^{K-1}.$$

Impulsive Action.

12. If impulsive forces be made to act in an incompressible fluid, or if impulsive pressures be excited by a sudden change of motion in a fluid mass, it can be shewn, exactly as in Articles 7 and 8, that the impulsive pressure at any point is the same in every direction, and that any impulsive pressure is transmitted equally through the fluid.

We may suppose, for instance, a closed vessel full of liquid, and an impulsive pressure P applied to it by means of a piston, area K, fitting in the side of the vessel; the impulsive pressures at all points will be the same, and will be measured by the quantity $\frac{P}{K}$.

13. PROP. *To find the relation between the impulsive pressure and the change of velocity.*

Imagine impulsive action transmitted in any way through a fluid. Let u, v, w, be the velocities at any point P, (x, y, z), immediately before the impulse, and u', v', w', immediately after, and let p be the impulsive pressure excited at P.

Suppose a small prism PQ, having its axis parallel to x, to be solidified, as in Art. 16; then, since the impulsive force at Q

$$= p + \frac{dp}{dx}\delta x,$$

$$-\kappa\frac{dp}{dx}\delta x = \kappa\rho\delta x\,(u' - u),$$

where κ is the sectional area,

$$\text{or}\quad \frac{dp}{dx} + \rho\,(u' - u) = 0.$$

Similarly,

$$\frac{dp}{dy} + \rho\,(v' - v) = 0,$$

$$\frac{dp}{dz} + \rho\,(w' - w) = 0;$$

and therefore

$$dp + \rho\,\{(u' - u)\,dx + (v' - v)\,dy + (w' - w)\,dz\} = 0.$$

EXAMPLE. *An open vessel containing fluid is suddenly moved upwards with a given velocity; it is required to find the impulsive pressure at any point.*

In this case, measuring z upwards from the base of the vessel, u, u', v, v', and w are zero;

$$\therefore\ dp + \rho w' dz = 0,$$

w' being the given velocity,

$$\text{or}\quad p + \rho w' z = C.$$

Let h be the height of the surface above the base; then $p = 0$, when $z = h$, and therefore

$$p = \rho w' (h - z),$$

$$\text{or}\quad p \propto \text{depth below the surface.}$$

MISCELLANEOUS EXAMPLES.

CHAPTERS I. II. III.

1. Two cylindrical vessels, containing different fluids, and standing near each other on a horizontal plane, are connected by a fine tube, which is close to the horizontal plane; when the communication is opened between them, determine which of the fluids will flow from its own vessel into the other, and find the condition that the equilibrium may not be disturbed.

In the latter case, determine the effect of placing equal weights of the same substance in each, its density being less than that of either.

2. One asymptote of a hyperbola lies in the surface of a fluid; find the depth of the centre of pressure of the area included between the immersed asymptote, the curve, and two given horizontal lines in the plane of the hyperbola.

3. A cone is totally immersed in a fluid, the depth of the centre of its base being given. Prove that, P, P', P'', being the resultant pressures on its convex surface, when the sines of the inclination of its axis to the horizon are s, s', s'', respectively,

$$P^2(s'-s'')+P'^2(s''-s)+P''^2(s-s')=0.$$

4. A rigid spherical shell is filled with homogeneous inelastic fluid, every particle of which attracts every other with a force varying inversely as the square of the distance; shew that the difference between the pressures at the surface and at any point within the fluid varies as the area of the least section of the sphere through the point.

5. A vessel in the form of a regular pyramid, whose base is a plane polygon of n sides, is placed with its axis vertical and vertex downwards and is filled with fluid. Each side of the vessel is moveable about a hinge at the vertex, and is kept in its place by a string fastened to the middle point of its base and to the centre of the polygon: shew that the tension of each

string is to the whole weight of the fluid as $1 : n \sin 2\alpha$, where α is the inclination of each side to the horizon.

6. A solid triangular prism, the faces of which include angles α, β, γ, is placed in any position entirely within an inelastic gravitating fluid: if P, Q, R, be the pressures on the three faces, which are respectively opposite to the angles α, β, γ, prove that

$$P \operatorname{cosec} \alpha + Q \operatorname{cosec} \beta + R \operatorname{cosec} \gamma$$

is invariable so long as the depth of the centre of gravity of the prism is unchanged.

7. A hemisphere is filled with heavy inelastic fluid; if the surface be divided by horizontal planes into n portions, on each of which the whole pressure is the same, and h_r be the depth of the r^{th} of these planes, prove that $h_r = \sqrt{\left(\frac{r}{n} \cdot a\right)}$, a being the radius of the hemisphere.

8. A hollow regular tetrahedron, very nearly filled with water, revolves about one of its edges at a vertical axis; compare the pressures on its faces.

9. An elliptic tube, half full of incompressible fluid, revolves about a fixed vertical axis in its own plane, with an angular velocity ω; the angle which the straight line joining the free surfaces of the fluid makes with the vertical, will be $\tan^{-1}\left(\frac{g}{p\omega^2}\right)$, where p is the perpendicular from the centre on the axis.

10. Find the centre of pressure of the area between the curve, $\sqrt{x} + \sqrt{y} = \sqrt{a}$, and the axes, the origin being in the surface.

11. Three equal cylinders are placed in contact on a horizontal plane, sufficiently rough to prevent sliding; find how much water must be poured into the space between the cylinders in order to disturb the equilibrium.

12. If the depths of the angular points of a triangle below

the surface of a fluid be a, b, c, shew that the depth of the centre of pressure below the centre of gravity is

$$\frac{(b-c)^2+(c-a)^2+(a-b)^2}{12(a+b+c)}.$$

13. Find the centre of pressure upon a portion of a vertical cylinder containing fluid, the portion being such as when unwrapped to form an isosceles triangle, the base of which when forming part of the cylinder is horizontal, and the vertex at the surface of the fluid. If this portion be divided into two equal parts by a vertical plane, find the least couple which will prevent either of the parts from turning round.

14. Two cubical vessels of height a have their bases horizontal and a common vertical face, in which an aperture is cut in the form of an equilateral triangle, whose vertex is in the base and opposite side horizontal, the length of the side being a. Fitted into this aperture is a prism of length l ($l < a$), which slides freely. Equal volumes of two fluids, the specific gravities of which are in the ratio 27 : 8, are poured into the respective vessels. Determine under what conditions the prism may be in equilibrium, and prove that it never can be so unless

$$l \text{ be} > \frac{2}{3}a.$$

15. A spherical shell, whose interior radius is a, is filled with fluid of uniform density ρ, and revolves with uniform angular velocity ω about the vertical diameter of the shell; shew that, if the total normal pressure on the upper half of the shell be to that on the lower half as $m : n$, the pressure at the highest point of the fluid is

$$\rho\left\{\frac{3m-n}{n-m}\frac{ga}{2}-\frac{\omega^2a^2}{3}\right\}.$$

16. A closed vessel full of fluid is made to revolve with uniform angular velocity ω about a vertical axis through its highest point; shew that the total pressure of the fluid on the surface of the vessel is increased by $\frac{1}{2}Ak^2\rho\omega^2$: A being the area of the sur-

face, k the radius of gyration of the surface about the vertical axis, and ρ the density of the fluid.

How may the total pressure be found, if the axis do not pass through the highest point of the surface?

17. Two very small spheres, of the same size but different densities, are connected by a fine string and immersed in a fluid, which rotates uniformly about a fixed axis, and is not acted upon by any forces; the density of the fluid being intermediate between the densities of the spheres, find their position of relative equilibrium.

18. A hollow sphere, filled with equal quantities of two incompressible fluids which do not mix, revolves uniformly about its vertical diameter, and the fluid particles are relatively at rest. Find the angular velocity when the lighter fluid just touches the lowest point in the surface of the sphere.

19. A conical vessel is divided into two parts by a plane through its axis, and the parts are prevented from separating by a string which is a diameter of the rim of the vessel and is perpendicular to the dividing plane, and by a hinge at the vertex. Supposing the vessel placed with its vertex downwards, its axis vertical, and to be filled with fluid, compare the tension of the string with the weight of the fluid.

20. All space being supposed filled with an elastic fluid whose volume at a given density is known, the particles of which are attracted to a given point by a force varying as the distance; find the pressure on a circular disk placed with its centre at the centre of force.

21. In a solid sphere two spherical cavities, the radii of which are each equal to half the radius of the solid sphere, are filled with fluid; the solid and fluid particles attract each other with forces which vary as the distance; shew that the surfaces of equal pressure are spheres concentric with the solid sphere.

22. A hollow cylinder is filled with inelastic fluid and made to revolve about a vertical axis attached to the centre of its upper plane face with a velocity sufficient to retain it at the

same inclination to the axis. Find at what point of the face a hole might be bored without loss of fluid.

23. A mass of inelastic fluid is contained between three co-ordinate planes, each of which attracts with a force varying as the distance, and the absolute forces of attraction μ, μ', μ'', are in harmonic progression. Half an ellipsoid is fixed with its plane face against one of the co-ordinate planes, and its surface touching the other planes, its axes being parallel to the co-ordinate axes and proportional to

$$\frac{1}{\sqrt{\mu}}, \frac{1}{\sqrt{\mu'}}, \frac{1}{\sqrt{\mu''}}.$$

If there be not sufficient fluid quite to cover the ellipsoid, the uncovered part will be bounded by a circle.

24. A uniform inelastic fluid contained in a closed vessel revolves about a fixed axis with a given angular velocity, and has in it a particle of solid matter which at a given epoch has a given position in the fluid and the same motion as the adjacent particles of fluid, find what its course in the fluid will be afterwards; neglecting the resistance. What will be the difference if the fluid and particle, instead of being contained in a closed vessel, be attracted to the axis of revolution by a force varying as the distance?

25. A mass of homogeneous fluid is subject to the mutual gravitation of its particles, and to a repulsive force tending from a plane through its centre of gravity and varying as the perpendicular distance from that plane; shew that the conditions of equilibrium will be satisfied if the surface be a prolate spheroid of a certain ellipticity, provided the repulsive force be not too great. *Smith's Prize Examination*, 1857.

CHAPTER IV.

26. A rod of length a and density ρ is moveable freely about one end, which is fixed at a depth c below the surface of a fluid of density σ; prove that the rod will remain at rest, when inclined to the vertical, provided that

$$\frac{\sigma}{\rho} > 1 \text{ and } < \frac{a^2}{c^2}.$$

Shew that such a position is one of stable equilibrium.

27. An inverted vessel formed of a substance which is heavier than water contains enough air to make it float; prove that if it be pushed down through a certain space, it will be in a position of equilibrium, which for vertical displacements will be a position of unstable equilibrium.

28. Two rods of the same substance have their ends fastened together at a given angle, and float in a heavy fluid with the angle immersed; shew that the problem of finding the positions of equilibrium is the same as that of drawing normals to a parabola from a point within it. Hence find the number of positions of equilibrium.

29. A hollow hemispherical cup is closed by a lid of the same small thickness and of the same substance; shew that, if it float in a liquid of known specific gravity with its centre in the surface, the inclination of the lid to the vertical will be $11° 15'$.

30. A hollow cylinder containing air is fitted with an air-tight piston which when the cylinder is placed vertically is at a given height above the base; the cylinder being now inverted and placed vertically in a fluid sinks partly below the surface; find the position of equilibrium.

31. If the height of a right circular cone be equal to the diameter of the base, it will float, with the slant side horizontal, in any fluid of greater specific gravity.

32. A conical shell floats in unstable equilibrium; how much water must be poured in to make the equilibrium stable?

33. A cone is placed in water, its axis being vertical and its vertex on the base of the vessel; find the least depth of water consistent with stable equilibrium.

34. Water rests upon mercury, and a cone is too heavy to rest without its vertex penetrating the mercury; find the density of the cone that the equilibrium may be stable.

35. A solid generated by the revolution of the curve $y \propto x^{\frac{n}{2}-1}$ around the axis of x, floats in a fluid with a portion h

of the axis immersed. If the solid be depressed through $(n^{\frac{1}{s-1}} - 1) h$, it will, on its return, just emerge from the fluid.

36. A right cone is floating with its axis vertical and vertex downwards in a fluid, and $\frac{1}{n}$th part of the axis is immersed; a weight equal to the weight of the cone is placed on the base, upon which the cone sinks till its axis is totally immersed, before rising, shew that

$$n^3 + n^2 + n = 7.$$

37. A thin cylinder, closed at the bottom, but open at the top, has a closely fitting piston without weight capable of moving in it without friction. If this cylinder be filled with atmospheric air, and, being just immersed in water with its axis vertical, be allowed to sink, find its velocity when it is at a given depth, the resistance being neglected.

38. A solid cone, whose axis is vertical and vertex downwards, is moveable about an axis coincident with a generating line; to what depth must the system be immersed in water, in order that the equilibrium of the cone may be stable?

39. A right prism on a square base has another prism, also on a square base, attached to it, so that their axes are coincident and sides parallel, and the whole floats on a fluid with their common plane in the plane of floatation. If the sides of the bases of the two prisms are in the ratio 2 : 1, find their limiting heights in order that the equilibrium may be stable.

40. A heavy cube is moveable about an axis, which passes through, and bisects, the opposite sides of one face; this axis being fixed horizontally within an empty vessel, so that the cube is suspended in the position of equilibrium, find the depth to which fluid must be poured in, so as to render the equilibrium unstable, and the greatest ratio of the densities of the cube and fluid, that this may be possible.

Supposing the cube half immersed and the equilibrium stable, find the time of a small oscillation.

41. A cylindrical diving-bell is suspended with its axis vertical at a depth such that the water rises half way up the bell: find the least distance of the centre of gravity of the bell from the centre of its upper surface, consistent with the condition that the equilibrium may be stable with reference to an angular displacement of the axis.

42. A cylinder makes vertical oscillations in a fluid contained in another cylinder, the radius of which is n times that of the former; shew that the depth of the axis immersed when in a position of rest is $gt^2n^2 \div \pi^2 (n^2 - 1)$ where t is the time of an oscillation.

43. A vessel in the form of a paraboloid with its axis vertical, contains a quantity of fluid equal in volume to that of a segment of a paraboloid, of the same latus rectum, floating in it: if this be raised till its vertex is just in the surface of the fluid, and if it then sink to a depth equal to $\frac{3}{4}$ of its axis before returning, shew that the density of the fluid : that of the paraboloid :: 48 : 7.

44. If a body float at rest, shew that for any displacement, consistent with the condition that the weight of the fluid displaced be equal to that of the float, the difference of the distances of the centres of gravity of the float and of the fluid displaced below the surface of the fluid will, in general, be a maximum or minimum according as the equilibrium is unstable or stable.

Moreover if Z be this difference, and the body be symmetrical with respect to a vertical plane, perpendicular to the line about which the displacement aforesaid is made, and θ be the inclination of any fixed line in the body and in that plane to the vertical, the time of a small oscillation will be that of a simple pendulum of which the length is $\dfrac{k^2}{\dfrac{d^2Z}{d\theta^2}}$, where k is the radius of gyration about a line through the centre of gravity parallel to the axis of displacement.

Mention any conditions which limit the generality of these theorems.

CHAPTERS V. VI. VII.

45. A quantity of air in a vessel will, at the zero temperature, support 30 inches of mercury; to what temperature must it be raised in order to support 31 inches?

46. A right cylindrical vessel on a plane base contains a certain quantity of elastic fluid, which is confined within it by a disc exactly similar and parallel to the base; shew that the pressure of the fluid on the curved surface of the cylinder is independent of the position of the disc, the temperature of the fluid being constant.

47. The pressure of a quantity of air, saturated with vapour, is observed; the mixture is then compressed into half its former volume, and, after the temperature has been lowered until it becomes the same as at first, the pressure is again observed; hence find what would be the pressure of the air (occupying its original space) if it were deprived of its vapour without having its temperature changed.

48. A closed vessel in the form of a right cone is placed with its base on a horizontal plane: supposing it to be filled with fluid through a small orifice at its vertex, prove that the horizontal tension of the vessel at any point varies as the area of the circular section through the point.

49. A heavy sphere is placed in a vertical cylinder, filled with atmospheric air, which it exactly fits. Find the density of the air in the cylinder when the sphere is in a position of permanent rest.

50. In the $(n+1)^{\text{th}}$ ascent of the piston in Smeaton's air-pump, find the position of the piston when the highest valve, (whose weight may be neglected), begins to open; and shew that in that position the tension of the piston rod : pressure of the atmosphere on the piston

$$:: 1-\left(\frac{A}{A+B}\right)^n : 1-\left(\frac{A}{A+B}\right)^n \cdot \frac{B}{A+B}.$$

51. Two barometers of the same length and equal transverse sections each contain a small quantity of air; their readings at one time are h, k, and at another time h', k'; compare the quantities of air in them.

52. A cylindrical tube, containing air, is closed at one extremity by a fixed plate, the other extremity being open; a piston just fitting the tube slides within it, and the centres of the plate and piston are connected by an elastic string, the modulus of elasticity of which is equal to the atmospheric pressure on the piston; prove that, if l be the natural length of the string, and a its length when the air between the piston and the fixed plate is in its natural state, l being less than a, the length of the string in the position of equilibrium will be $(la)^{\frac{1}{2}}$. If the piston be slightly displaced from this position, find the time of a small oscillation.

53. A mass of elastic fluid is confined within a hollow sphere, and repelled from the centre of the sphere by a force $\frac{\mu}{r}$; shew that the whole pressure on the sphere : the pressure which would be exerted if no force were acting :: $3k + \mu : 3k$.

54. A piston without weight fits into a vertical cylinder, closed at its base and filled with air, and is initially at the top of the cylinder; if water be slowly poured on the top of the piston, shew that the upper surface of the water will be lowest when the depth of the water is $\sqrt{(ah)} - h$, where h is the height of the water barometer, and a the height of the cylinder.

55. An elastic spherical envelop is in equilibrium when it contains air at twice the atmospheric density, and its radius is twice the natural size; if the barometer fall $\frac{1}{n}$th of an inch, find the time of a small oscillation in the magnitude of the envelope.

56. A straight tube, closed at one end and open at the other, revolves with an uniform angular velocity about an axis meeting the tube at right angles; neglecting the action of gravity, determine the law of density of the air within the tube, and compare the pressure at the closed end with the atmospheric pressure.

57. A bent tube of uniform bore, whose arms are at right angles to one another, revolves with uniform angular velocity ω

about the axis of one of its arms, which is vertical and has its extremity immersed in water. Prove that the height to which the water will rise in the vertical arm is

$$\frac{\Pi}{g\rho}\left(1 - \epsilon^{-\frac{\omega^2 a^2}{2k}}\right),$$

where a is the length of the horizontal arm, Π the atmospheric pressure, and ρ the density of water.

58. If the earth be supposed spherical and covered with an ocean of small depth, and if the attraction of the particles of water on each other be omitted, the ellipticity of the ocean spheroid will be given by the equation

$$2\epsilon = \frac{\text{centrifugal force at the equator}}{\text{force of gravity at the earth's surface}}.$$

CHAPTERS VIII.—XIII.

59. The main of the water supply of a town is one foot in diameter, and side pipes four inches in diameter leave it at intervals successively; find the velocity in the main after any number of these side pipes have been passed; that before passing any of them being given, and the water being supposed to flow freely and steadily in all.

60. A closed cubical box, very nearly filled with fluid, is placed on a smooth horizontal table, so that two of its faces are parallel to the edge of the table, and a string, passing over the edge and supporting a weight, is fastened to the middle point of the base of the nearest face; determine the surfaces of equal pressure in the fluid during the motion, and compare the pressures on the top and on the base of the box.

If a small sphere of greater density than the fluid be suspended in it by a string fastened to the top of the box, and another small sphere of less density than the fluid be attached by a string to the base of the box, find the directions of the strings when the spheres are in equilibrium relative to the fluid.

What would be the effect on these spheres of suddenly destroying the motion of the box?

61. Two cylindrical vessels of equal height containing water are suspended with their axes vertical to the ends of a string

passing over a fixed smooth pulley in a vertical plane; neglecting the weights of the vessels, shew that the whole pressures, during the motion, on the curved surfaces of the cylinders are inversely in the ratio of their radii, and that the pressures on their bases are equal.

62. On a horizontal plane stand a frustum of a cone filled with fluid and an empty hemispherical bowl. A small orifice is made in the former, in the vertical plane through the axes of both vessels, and at a height equal to a radius of the latter. Given that the bowl stands altogether within the range of the stream as it first issues, determine the quantity of fluid which will be poured into it.

63. A sphere of less specific gravity than water is placed at a given depth in a stream running with a given uniform velocity, and then left to the action of the stream; find the motion and path of the sphere.

64. In the eruption of a volcano, it has been remarked that every ejection of stones is accompanied with an explosion like artillery, when heard at a distance, but that when near, the sound resembles rather that of a loud and deep sigh, unaccompanied by any sudden burst; explain this phenomenon, and apply the principle of your explanation to shew that a flash of lightning may produce a lengthened peal of thunder, without the reverberation of the clouds.

65. If there be vibrating plates at each end of a tube, how must the times of vibration be related, so that musical notes may be produced?

66. The same musical note is produced in each of two railway trains travelling in the same direction; shew that a person standing on the line between the trains will hear beats, and determine the number of beats per second, when the pitch of the note and the velocities of the trains are given.

67. A uniform bent tube of given length, the two legs of which are vertical and of the same height, is filled with fluid. A heavy plug exactly fitting the tube is placed upon the surface

of the fluid in one of the open ends of the tube, and is allowed to descend by its own weight; determine the greatest depth to which it will sink; and if the length and weight of the plug be small, shew that it will displace very nearly twice its own weight of the fluid.

In the latter case determine the amplitude and time of an oscillation.

68. A globule of inelastic fluid falls under gravity through a medium, which exerts on it at any point of its surface a pressure which is equal to a constant pressure increased, if the surface at that point be moving in the direction opposite to that in which the pressure acts, and diminished, if in the same direction, by a quantity proportional to the normal velocity of the surface at that point; shew that when the globule has attained its terminal velocity its figure will be a sphere.

69. A circular cricket-field is surrounded by a fence consisting of a great number of equidistant palings, the plane of each being perpendicular to the radius; a player at the centre upon striking a ball hears an echo of the sound of the blow, while to a spectator placed elsewhere the echo is replaced by a metallic ringing sound: explain this.

70. At a station on a railway passed at full speed by a train, a certain musical note is sounded; explain the difference of the sounds heard by a person in the train as it approaches to and recedes from the station.

71. A cylindrical aperture cut through a solid cylinder, with its axis parallel to that of the cylinder, is filled with fluid of the same density as the cylinder, and closed so that no fluid can escape; the cylinder being made to roll on a rough horizontal plane, shew that its motion will be uniform.

Determine also the surfaces of equal pressure at any instant, and trace their changes through a whole revolution of the cylinder.

72. Investigate the propagation of a sound in a vertical tube, shewing that its velocity will be the same at all points

in the tube, the temperature being supposed uniform to begin with.

73. A mass of homogeneous incompressible fluid subject to no external force but gravity is in motion; prove that the particles cannot describe circles about a common vertical axis unless the surfaces of equal velocity be cylinders, and find an expression for the angular velocity of each cylinder if all surfaces of equal pressure are spheres.

74. A vertical cylindrical vessel, open at the top and containing water, is let fall from a given height on a horizontal plane; the vessel being inelastic, find the impulsive pressure at any point of the fluid at the instant of impact.

If a piece of cork be immersed in the fluid and be kept under the surface by a string fastened to the base of the vessel, find the impulsive tension of the string.

75. A vessel of given capacity, in the form of a surface of revolution with two circular ends, is just filled with inelastic fluid which revolves about the axis of the vessel, and is supposed to be free from the action of gravity: investigate the form of the vessel that the whole pressure which the fluid exerts upon it may be the least possible, the magnitudes of the circular ends being given.

Shew that, for a certain relation between the radii of the circular ends, the generating curve of the surface is the common catenary.

76. A vertical tube, open at both ends and of the same transverse section throughout, is kept at a uniform temperature: supposing the increase of temperature of any portion of air within the tube to be proportional to the time, shew that the velocity of the current of air at a distance x from the bottom is given by the equation

$$\frac{v^2 + 2gx}{2gc_1} = \log \frac{gv + ka}{c_2} + \frac{ka}{gv + ka}.$$

How may the constants c_1, c_2 be determined?

77. When the motion of an incompressible fluid is entirely in the direction of one plane and is steady, prove that, if $U = C$ be the equation to a system of lines of motion,

$$\frac{dU}{dy}\cdot\frac{d}{dx}\left(\frac{d^2U}{dx^2}+\frac{d^2U}{dy^2}\right)=\frac{dU}{dx}\cdot\frac{d}{dy}\left(\frac{d^2U}{dx^2}+\frac{d^2U}{dy^2}\right).$$

78. The portions Ax, Ax', of a stretched string xAx' are of different thicknesses; prove that any small transversal vibrations travelling from x to A will, on arriving at A, be partly reflected and partly transmitted to Ax', and that the displacements due to the incident, reflected, and transmitted vibrations are to each other as $1+\mu : 1-\mu : 2$, where μ is the ratio of the velocity of propagation in Ax to that in Ax'.

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